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Managing Financial Crises *

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Abstract

In this paper, we revisit the question of how to manage financial crises using the framework proposed by Bianchi and Mendoza (2018). We show that this model economy exhibits a multiplicity of constrained-efficient equilibria, which arises because the private shadow value of collateral influences the forward-looking asset price. Among these equilibria, the specific one studied by Bianchi and Mendoza (2018) can be implemented using a tax/subsidy on debt alone. In that case, both the ex ante tax and ex post subsidy are quantitatively important for welfare under the optimal time-consistent policy. Limiting either component can lead to a welfare loss relative to the unregulated competitive equilibrium, highlighting the complementarity between crisis prevention and crisis resolution tools. We also show that, under certain conditions, all Pareto-dominant constrained-efficient equilibria entail the unconstrained allocation chosen by a social planner subject to the country budget constraint, and this allocation can be implemented with purely ex post policies.

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1 Introduction

How to manage financial crises remains one of the central questions for policymakers and scholars. A time-honored perspective, dating back to Bagehot (1873), argues that crises can be entirely managed ex post by providing liquidity to solvent institutions against good collateral and at penalty rates. The experience of the global financial crisis, however, showed that crisis resolution ex post can be extremely costly and that bailouts raise concerns about moral hazard and time inconsistency. In response, both policy and research shifted towards preventing crises ex ante, giving rise to a large literature on macroprudential regulation. While the case for ex ante policies is now well established, less is known about how they interact with ex post crisis management policies.

In this paper, we address this question within the workhorse model of financial crises developed by Bianchi and Mendoza (2018), building on Mendoza (2010). In this environment, crises are triggered by an occasionally binding collateral constraint that amplifies shocks through asset prices and borrowing capacity. Our focus is normative. What should a time-consistent planner do in such an environment? And what set of policies can implement the optimal allocation?

Our first contribution is to show that multiple constrained-efficient Markov perfect equilibria (MPE) can exist, and they are ranked by welfare. The source of the multiplicity is the forward-looking nature of the asset price, which depends on the *private* agents' future shadow value of collateral. We show that the current planner, taking the next-period decision rules as given, is indifferent between choosing any nonnegative private shadow value when the constraint binds. In an MPE, however, the current planner's choice must be consistent with the decision rule of the future planner, and different rules for the private Lagrange multiplier imply different equilibrium asset prices and constrained-efficient allocations. Among these, we identify a welfare-dominant MPE that entails the "unconstrained allocation"—the allocation a social planner would choose if the collateral constraint did not exist. In this case, the only remaining friction is market incompleteness, so the unconstrained allocation is first-best given the financial structure. From a policy perspective, this result highlights the importance of striving to eliminate crises altogether.

The MPE multiplicity we identify is related to the problem of determining prices in planning problems with financial constraints. Kehoe and Levine (1993) showed that conditional efficiency and constrained efficiency—that differ in whether spot prices are held fixed when considering deviations from a given allocation—are, in general, neither necessary nor sufficient for each other. Following Lorenzoni (2008), the literature on economies with collateral constraints has typically adopted the notion of constrained efficiency, where the planner selects allocations and prices subject to the requirement that prices satisfy the optimality conditions of the competitive equilibrium. In line with this approach, we also focus on constrained-efficient allocations.

Earlier studies do not find the MPE multiplicity because prices are determined by static optimality conditions (Lorenzoni, 2008; Bianchi, 2011; Benigno et al., 2013; Dávila and Korinek, 2018; Ottonello et al., 2022), private borrowing is constrained by aggregate collateral (Jeanne and Korinek, 2019), or an additional constraint is imposed on the planning problem that makes the equilibrium unique (Bianchi and Mendoza, 2018). Moreover, the MPE multiplicity is fundamentally different from the multiplicity of *competitive* equilibria that can arise due to nonlinearities introduced by financial constraints (Bocola and Lorenzoni, 2020; Jeanne and Korinek, 2020; Schmitt-Grohé and Uribe, 2021). The MPE multiplicity reflects a policy implementation indeterminacy rather than an equilibrium selection problem in a decentralized market.

After characterizing the set of constrained-efficient MPE, we show how these equilibria can be implemented as regulated competitive equilibria. Specifically, we demonstrate that a combination of taxes or subsidies on debt and intermediate inputs is sufficient to implement the entire set of MPE. Furthermore, we show that the tax on debt has two components: an ex ante component, which is active when the collateral constraint is slack and serves a macroprudential role, and an ex post component, which becomes active when the constraint binds and supports crisis resolution. The tax on intermediate inputs is required only in the presence of a working capital constraint, and only when the collateral constraint binds.

Within the set of constrained-efficient MPE, there is one particular equilibrium—the one studied by Bianchi and Mendoza (2018)—that can be implemented using only the tax or subsidy on debt. In this case, the planner imposes an additional restriction on the agent's shadow value of collateral, ensuring it satisfies the private optimality condition for the choice of intermediate inputs. This leaves the planner with a single degree of freedom to improve upon the competitive allocation, namely the choice of bond holdings. As a result, a tax or subsidy on debt alone is sufficient to implement this *specific* MPE, although it is not welfare-dominant within the constrained-efficient set.

Quantitatively, implementing this particular MPE entails a nonnegative debt tax levied in normal times (ex ante component) and a *subsidy* provided in bad times (ex post component). The magnitude of the ex post debt subsidy provided during a typical financial crisis is more than four times larger than the magnitude of the macroprudential tax levied before the crisis. However, both components are essential for welfare gains from the optimal policy. To demonstrate the latter point, we compute restricted optimal time-consistent policies subject to a lower and/or upper bound on the debt tax and show that restricting either component can lead to welfare losses of comparable magnitude. In particular, a purely macroprudential time-consistent policy that discourages borrowing in good times but lacks ex post interventions during financial crises generates welfare losses. The two components of the optimal policy are thus complementary to each other.

The complementarity between the ex ante and ex post components of the debt tax is consistent with previous findings that emphasize the joint use of ex ante and ex post tools, including Benigno et al. (2016) on the optimality of combining prudential capital controls with ex post exchange rate interventions during crises, Jeanne and Korinek (2020) on macroprudential bank regulation paired with crisis-time liquidity provision, and Iacoviello et al. (2025) on Pareto improvements from policy rules that tax housing investment in good times and subsidize it in bad times. The novelty of our contribution lies in retrieving this result for the same instrument (i.e., tax on debt) used optimally both ex ante and ex post, which is important for the design of the appropriate regulatory framework in practice.

Finally, we compute the optimal time-consistent policy that implements the welfare-dominant unconstrained allocation. This policy includes taxes on both debt and intermediate inputs. In this case, the ex ante component of the debt tax is zero, and the optimal debt market intervention consists solely of an ex post debt subsidy—a result we also establish analytically. The debt subsidy adjusts in response to the agent's shadow value of collateral, offsetting the distortion introduced by the collateral constraint. At the same time, the subsidy on intermediate inputs addresses the inefficiency in input choices caused by the working capital constraint. As a result, agents accumulate significantly more debt in the regulated competitive equilibrium than in the unregulated equilibrium, indicating that the unregulated equilibrium features underborrowing.

The result that, in the absence of the working capital constraint, an ex post debt subsidy alone is sufficient to implement the welfare-dominant allocation is consistent with findings in Benigno et al. (2016), Jeanne and Korinek (2020), and Benigno et al. (2023), who show that if ex post interventions are available and costless, there is no need for ex ante (macroprudential) regulation. Similarly, Bocola and Lorenzoni (2020) demonstrate that ex post support for banks in a financially dollarized economy can eliminate self-fulfilling financial crises. The critical novelty of our result lies in deriving it in a setting with an individual, asset-price-based collateral constraint. In contrast to existing work that focuses on income-based constraints, aggregate constraints, or nominal rigidities, our framework shows that even when borrowing is limited by individual collateral positions—tied directly to market-determined asset prices—a purely ex post policy can be sufficient to fully restore efficiency. This extends the policy insight to a broader class of financial frictions and planning problems.

Returning to Bagehot's dictum of the late 19th century and the macroprudential policy revolution of the early 21st century, our analysis underscores that effective financial crisis management requires macroprudential regulation and crisis resolution to be designed jointly, rather than in isolation. The key policy message of our analysis is to emphasize the role of lender-of-last-resort interventions as crucial not only for managing crises ex post but also for enabling effective macroprudential policy. Indeed, in the absence of ex post interventions, preventive tools alone can be costly and may even reduce welfare. The right balance between crisis prevention and crisis resolution cannot be determined independently of the institutional and economic structure. Instead, it must reflect the specific constraints and feedback effects that shape the dynamics of the financial crisis.

The rest of the paper is structured as follows. Section 2 presents the model economy and its competitive equilibrium. Section 3 discusses efficiency in this model environment. Section 4 characterizes the optimal time-consistent policies that implement the constrained-efficient equilibria. Section 5 quantitatively assesses the sources of welfare gains from the optimal policy. Section 6 concludes. Appendices comprise proofs and computational details.

2 Model

In this section, we briefly describe the model economy and define its decentralized competitive equilibrium. We refer the reader to Bianchi and Mendoza (2018) for a comprehensive description.

Consider an infinite-horizon small open economy in discrete time. The economy's exogenous state is $s_t = (z_t, R_t, \kappa_t)$, where $z_t \in [\underline{z}, \overline{z}]$ is the total factor productivity (TFP), $R_t \in [\underline{R}, \overline{R}]$ is the gross interest rate, and $\kappa_t \in [\underline{\kappa}, \overline{\kappa}]$ is the credit regime specified below. Let $S = [\underline{z}, \overline{z}] \times [\underline{R}, \overline{R}] \times [\underline{\kappa}, \overline{\kappa}]$. We assume that $S \subset \mathbb{R}^3_{++}$ is finite and $\{s_t\}_{t=0}^{\infty}$ is a stationary Markov process. We denote the histories of states as $s^t = (s_0, s_1, \ldots, s_t) \in S^t$, where $S^t = S^{t-1} \times S$ for all $t > 0$ with $S^0 = \{s_0\}$. The conditional expectation operator given a specific s^t is $\mathbb{E}_t$. To simplify notation, whenever possible, the history dependence is implicit.

There is a unit measure of domestic agents that are firm-households.[1] The representative agent's preferences over history-contingent sequences of consumption c_t and labor h_t are described by the utility function

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t - g(h_t)), \tag{1}$$

where $\beta \in (0, 1)$ is the discount factor, $u : \mathbb{R}_+ \to \mathbb{R}$ is a twice continuously differentiable,

[1] Bianchi and Mendoza (2018, Appendix C) show that there exists an equivalent environment with separate households and firms.

strictly increasing, and strictly concave period utility function that satisfies $\lim_{x \downarrow 0} u'(x) = \infty$, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a twice continuously differentiable, strictly increasing, and convex labor disutility function. We assume $\beta \overline{R} < 1$, so that the equilibria we consider have well-defined stationary distributions. The GHH (Greenwood et al., 1988) preferences over a composite good $\tilde{c}_t \equiv c_t - g(h_t)$ eliminate the impact of the variations in marginal utility of consumption on the labor supply. Although this assumption can be relaxed, it significantly simplifies the theoretical analysis.

The agent produces the final good from capital k_t, labor, and intermediate inputs v_t using a concave Cobb—Douglas production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$. The initial level of capital is $k_0 = 1$. The price of capital is q_t, while the price of internationally traded inputs is fixed at $p_v > 0$. The agent can invest in a one-period bond b_{t+1} traded internationally at the price $1/R_t$, with $b_0 \in B = [\underline{b}, \overline{b}] \subset \mathbb{R}$ given. The agent's budget constraint is thus

$$c_t + q_t k_{t+1} + \frac{b_{t+1}}{R_t} \leq z_t F(k_t, h_t, v_t) - p_v v_t + q_t k_t + b_t. \tag{2}$$

The agent can issue debt ($b_{t+1} < 0$) and must finance a fraction $\theta \in [0, 1]$ of intermediate inputs in advance with an intraperiod loan at a zero interest rate. Borrowing requires collateral in the form of the capital stock, but only a fraction κ_t of the value of capital is pledgeable. Consequently, the agent faces a collateral constraint[2]

$$-\frac{b_{t+1}}{R_t} + \theta p_v v_t \leq \kappa_t q_t k_t. \tag{3}$$

The agent's problem is to choose $\{(c_t, h_t, v_t, b_{t+1}, k_{t+1})\}_{t=0}^{\infty}$ to maximize (1) subject to (2) and (3) for all (t, s^t). The first-order conditions for this problem are

$$g'(h_t) = z_t F_h(k_t, h_t, v_t), \tag{4}$$

$$\left(1 + \theta \frac{\mu_t}{u'(\tilde{c}_t)}\right) p_v = z_t F_v(k_t, h_t, v_t), \tag{5}$$

$$u'(\tilde{c}_t) = \beta R_t \mathbb{E}_t u'(\tilde{c}_{t+1}) + \mu_t, \tag{6}$$

$$q_t u'(\tilde{c}_t) = \beta \mathbb{E}_t \left[u'(\tilde{c}_{t+1}) \Big(z_{t+1} F_k(k_{t+1}, h_{t+1}, v_{t+1}) + q_{t+1} \Big) + \mu_{t+1} \kappa_{t+1} q_{t+1} \right], \tag{7}$$

$$0 = \mu_t \left(\kappa_t q_t k_t + \frac{b_{t+1}}{R_t} - \theta p_v v_t \right), \qquad \mu_t \geq 0, \tag{8}$$

where μ_t is the Lagrange multiplier on the collateral constraint (3). The equation (4) equates the marginal rate of substitution of leisure for consumption with the marginal product of

[2] Bianchi and Mendoza (2018, Appendix A.5) provide a microfoundation through a limited enforcement problem.

labor. According to (5), the working capital and collateral constraints introduce a wedge between the marginal product of intermediate inputs and their price. When the collateral constraint binds, and as long as $\theta > 0$, an increase in inputs must be compensated by a decrease in borrowing, thus raising the marginal cost of inputs compared to the case of no working capital constraint ($\theta = 0$). (6) is a standard bond Euler equation: when the borrowing constraint binds, other things equal, the agent's marginal utility of consumption today is greater than in the unconstrained case. At the same time, the collateral constraint introduces an additional marginal benefit of capital, since greater capital allows to borrow more when the constraint binds. This is captured in the asset pricing condition (7): if we solve it forward, we can express the asset price q_t as an expected discounted sum of dividends, where the discounting is adjusted to include the collateral value. Finally, (8) comprises the complementary slackness conditions associated with the collateral constraint (3).

The capital stock is in fixed supply normalized to one. We define a decentralized competitive equilibrium (**DE**) as follows.

Definition 1 (Competitive equilibrium)**.** *A decentralized competitive equilibrium is an allocation* $\{(c_t, h_t, v_t, b_{t+1}, k_{t+1})\}_{t=0}^{\infty}$, *prices* $\{q_t\}_{t=0}^{\infty}$, *and Lagrange multipliers* $\{\mu_t\}_{t=0}^{\infty}$, *such that the following holds.*

1. *Given prices, the allocation solves the agent's problem: that is, together with Lagrange multipliers, it satisfies (2) holding with equality and (3)–(8) for all* $t \geq 0$ *and* $s^t \in S^t$.

2. *Prices are such that the capital market clears:* $k_{t+1}(s^t) = 1$ *for all* $t \geq 0$ *and* $s^t \in S^t$.

Going forward, it will be useful to represent the DE using the recursive notation. We denote the aggregate state as $x = (b, s) \in X = B \times S$, where $b \in B$ is aggregate bond holdings, and $s = (z, R, \kappa) \in S$ is the exogenous state. We denote the conditional expectation operator given $s \in S$ as $\mathbb{E}_s$ and use $y(x)$ interchangeably with y_x to denote the value of a variable y at the state $x \in X$. Let $\operatorname{int} Y$ and $\operatorname{cl} Y$ denote the interior and closure, respectively, of a generic set Y. Clearly, $B = [\underline{b}, \bar{b}]$ needs to be large enough, so that $b_{t+1}(s^t) \in \operatorname{int} B$ for all $t \geq 0$ and $s^t \in S^t$, although $\underline{b}$ cannot be too low due to the collateral constraint. An admissible B can be found numerically, given a specific model calibration.

Let $\mathcal{F}(X)$ denote the set of all real-valued functions on X. Imposing the capital market clearing condition in (2)–(8), we obtain the following recursive representation of the DE.

Remark 1 (Recursive equilibrium)**.** *A DE of Definition 1 is, equivalently, a set of allocation functions* $\{\tilde{c}, h, v, b'\} \subset \mathcal{F}(X)$, *an asset price function* $q \in \mathcal{F}(X)$, *and a Lagrange multiplier* *function $\mu \in \mathcal{F}(X)$ that satisfy*

$$\tilde{c}_x + \frac{b'_x}{R} = zF(1, h_x, v_x) - p_v v_x - g(h_x) + b, \tag{9}$$

$$-\frac{b'_x}{R} + \theta p_v v_x \leq \kappa q_x, \tag{10}$$

$$g'(h_x) = zF_h(1, h_x, v_x), \tag{11}$$

$$\left(1 + \theta \frac{\mu_x}{u'(\tilde{c}_x)}\right) p_v = zF_v(1, h_x, v_x), \tag{12}$$

$$u'(\tilde{c}_x) = \beta R \mathbb{E}_s u'(\tilde{c}_{x'}) + \mu_x \qquad \text{if } b'_x \in \operatorname{int} B, \tag{13}$$

$$q_x u'(\tilde{c}_x) = \beta \mathbb{E}_s \left[u'(\tilde{c}_{x'}) \Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'} \Big) + \mu_{x'} \kappa' q_{x'} \right], \tag{14}$$

$$0 = \mu_x \left(\kappa q_x + \frac{b'_x}{R} - \theta p_v v_x \right), \qquad \mu_x \geq 0, \tag{15}$$

with $x' = (b'_x, (z', R', \kappa'))$, for all $x = (b, s) = (b, (z, R, \kappa)) \in X$.

3 Efficiency Analysis

This section explores the properties of efficient and constrained-efficient allocations in the model environment described in Section 2.

We begin by defining the "unconstrained allocation" chosen by a benevolent social planner subject to the country budget constraint. This allocation is the first best given the existing financial markets. We show that the DE has two distortions relative to the unconstrained allocation that arise due to the collateral constraint. First, provided the collateral constraint may bind in some states, the DE features inferior consumption smoothing. Second, under the working capital constraint, the DE entails an inefficiently low level of intermediate inputs.

We then define a time-consistent constrained-efficient allocation as part of a Markov perfect equilibrium of a noncooperative game between successive benevolent social planners who face the same constraints as the representative agent but internalize the impact of allocations on the market price of capital. We argue that constrained-efficient equilibria are generally not unique in this model economy, and there may exist a Markov perfect equilibrium that entails the unconstrained allocation. Finally, if we drop the Markov perfection requirement and allow the planner's decisions to be history-contingent, we find that, under certain conditions, the unconstrained allocation is the unique constrained-efficient allocation, and any constrained-efficient plan is time consistent.

3.1 Unconstrained allocation

In this economy, the efficient allocation is the allocation chosen by a benevolent social planner subject to the country budget constraint (9). Since international financial markets are incomplete, the efficient allocation may be considered a *constrained optimum* (Diamond, 1967).[3] On the other hand, this allocation is the *first best* conditional on the existing financial markets—that is, the market for a one-period noncontingent bond (Itskhoki and Mukhin, 2023). We call this allocation the "unconstrained allocation" to emphasize that the planner is not subject to the collateral constraint. We define the unconstrained allocation in recursive form as follows.

Definition 2 (Unconstrained allocation)**.** *The unconstrained allocation is a set of functions* $\{\tilde{c}, h, v, b'\} \subset \mathcal{F}(X)$ *generated by the solution to the Bellman equation*

$$V(b, s) = \max_{\hat{c},\hat{h},\hat{v},\hat{b}} \Big[u(\hat{c}) + \beta \mathbb{E}_s V(\hat{b}, s') \Big],$$

subject to

$$\hat{c} + \frac{\hat{b}}{R} \leq zF(1, \hat{h}, \hat{v}) - p_v \hat{v} - g(\hat{h}) + b,$$

for all $(b, s) = (b, (z, R, \kappa)) \in X$*.*

Note that since the unconstrained problem in Definition 2 is not affected by the collateral constraint, κ is a redundant exogenous state, but we keep it as an element of s for consistency with other sections. We obtain the following characterization of the unconstrained allocation.

Proposition 1 (Unconstrained allocation)**.** *Let* $f(z) = \max_{\hat{h},\hat{v}}\{zF(1, \hat{h}, \hat{v}) - p_v \hat{v} - g(\hat{h})\}$*. Suppose* $\underline{b} > -\frac{\overline{R}}{\overline{R}-1} f(\underline{z})$ *if* $\overline{R} > 1$ *and* $\underline{b} < \frac{\underline{R}}{1-\underline{R}} f(\underline{z})$ *if* $\underline{R} < 1$*. Then there exists a unique solution to the Bellman equation in Definition 2. The allocation functions* $\{\tilde{c}, h, v, b'\} \subset \mathcal{F}(X)$ *are continuous and satisfy* (9), (11),

$$p_v = zF_v(1, h_x, v_x), \tag{16}$$

$$u'(\tilde{c}_x) = \beta R \mathbb{E}_s u'(\tilde{c}_{x'}) \qquad \text{if } b'_x \in \operatorname{int} B, \tag{17}$$

with $x' = (b'_x, (z', R', \kappa'))$*, for all* $x = (b, s) = (b, (z, R, \kappa)) \in X$*. The function* $\tilde{c}$ *is strictly increasing in* b*, and* b' *is strictly increasing in* b *whenever* $b'_x \in \operatorname{int} B$*. The functions* h *and* v *depend on* z *only, and* h*,* v*, and* f *are strictly increasing in* z*.*

Proof. See Appendix A.1. ∎

[3] If the domestic economy had access to a complete set of Arrow securities, the efficient allocation would entail the perfect consumption risk sharing between the domestic economy and the rest of the world.

Proposition 1 first provides restrictions on the admissible set of bond holdings $B = [\underline{b}, \overline{b}]$. In particular, if $\overline{R} > 1$, the lower bound $\underline{b}$ cannot be below than minus the natural borrowing limit $\frac{\overline{R}}{\overline{R}-1} f(\underline{z})$. Since the choice of labor and intermediate inputs is static, the planner's problem is similar to an income fluctuation problem (Schechtman and Escudero, 1977) with a stochastic endowment $f(z)$, where f is strictly increasing, and a stochastic interest rate R. Our assumption $\beta \overline{R} < 1$ ensures the existence of a well-defined ergodic distribution of bond holdings, which can be shown analytically under further assumptions on u, R, and z, or otherwise verified numerically.[4] This means that the bond holdings upper bound $\overline{b}$ can be set to a sufficiently big number, such that it never binds.

According to Proposition 1, the unconstrained labor h and intermediate inputs v are jointly defined by (11) and (16). Hence, h and v are independent of bond holdings b and only vary with the TFP z. Moreover, both h and v, and thus output $zF(1, h(z), v(z))$, are strictly increasing in z. Unconstrained next-period bond holdings b' are pinned down by the Euler equation (17) whenever $b'_x > \underline{b}$, in which case b' is strictly increasing in current bond holdings b.[5] (The absence of the collateral constraint is crucial for this fact.) The net consumption function $\tilde{c}$ is also strictly increasing in b.

Proposition 1 implies that the DE of Remark 1 has two distortions, both due to the collateral constraint. Both the DE and unconstrained allocations share the same labor optimality condition (11). Comparing (12) and (16), we observe that, if there is a working capital constraint ($\theta > 0$), intermediate inputs are inefficiently low in the DE whenever the collateral constraint is strictly binding ($\mu_x > 0$). In turn, lower inputs lead to lower labor and output. At the same time, if the collateral constraint is slack, labor and inputs in the DE are efficient, since they are jointly defined by (11) and (16), as in the unconstrained allocation. Moreover, (13) and (17) imply that net consumption is inefficiently low in the DE whenever $\mu_x > 0$, indicating inferior consumption smoothing. If the collateral constraint is *never* binding in the DE, its allocation is efficient, since in that case (12) is equivalent to (16) and (13) is equivalent to (17).

Remark 2 (Unconstrained equilibrium)**.** *Consider a DE of Remark 1. If $\mu_x = 0$ for all $x \in X$, the DE entails the unconstrained allocation of Definition 2.*

[4] If u has constant relative risk aversion form and R is deterministic, a sufficient condition is that z is either independent and identically distributed (Schechtman and Escudero, 1977) or $z \in \{\underline{z}, \overline{z}\}$ with $\Pr(z = \overline{z} \mid z = \overline{z}) \geq \Pr(z = \overline{z} \mid z = \underline{z})$ (Huggett, 1993).

[5] Numerically, if $\underline{b}$ is sufficiently close to minus the natural borrowing limit, we have $b'_x > \underline{b}$ for all $x \in X$ except $x = (\underline{b}, (\underline{z}, \overline{R}, \cdot))$, given an arbitrarily fine grid for B. Moreover, $b_t > \underline{b}$ for all t over a 100,000-period stochastic simulation after a 1,000-period burn-in.

3.2 Constrained-efficient equilibria

In general, policy constraints may prevent the possibility of implementing the efficient allocation in a competitive equilibrium. For this reason, an alternative concept of *constrained* efficiency has received significant attention. Constrained efficiency in environments with price-dependent borrowing constraints has been studied, for instance, by Kehoe and Levine (1993), Caballero and Krishnamurthy (2001), Lorenzoni (2008), and Dávila and Korinek (2018). In turn, this theory is related to the earlier analysis of constrained efficiency in incomplete-market economies by Diamond (1967), Hart (1975), Stiglitz (1982), and Geanakoplos and Polemarchakis (1985). A unified treatment was given by Farhi and Werning (2016).

A constrained-efficient allocation is an allocation chosen by a social planner (**SP**) that faces the same constraints as the representative agent but internalizes the impact of allocations on the market price of capital. Hence, such a social planner maximizes the agent's utility subject to the country budget constraint (9), collateral constraint (10), and asset pricing constraint (14). In addition, since the planner faces the same collateral constraint as the agent, and the market price of capital depends on whether the collateral constraint is binding in the future, as reflected by the Lagrange multiplier $\mu_{x'}$ affecting the next-period payoff on capital in (14), any SP allocation that can be implemented in a competitive equilibrium with some government policies must satisfy the agent's complementary slackness conditions (15).

Since the asset pricing constraint (14) is forward-looking, the constrained-efficient allocation under commitment is generally *time inconsistent*. For example, by committing to a higher future marginal utility of consumption (lower future consumption), the planner can increase the current asset price through the stochastic discount factor (SDF) and relax the collateral constraint in the binding state. When future arrives, however, a higher *current* marginal utility (lower current consumption) implies a lower current asset price, once again through the SDF, and thus the collateral constraint may bind or become more binding, which is suboptimal if the past promises need not be kept. Our focus is going to be on time-consistent equilibria.

We next define a constrained-efficient allocation as part of a *Markov perfect equilibrium* (**MPE**) of a noncooperative policy game between successive benevolent social planners (Maskin and Tirole, 1988, 2001; Krusell et al., 1996; Klein et al., 2008).

Definition 3 (Constrained-efficient MPE). *A constrained-efficient MPE is a set of allocation functions $\{\tilde{c}, h, v, b'\} \subset \mathcal{F}(X)$, an asset price function $q \in \mathcal{F}(X)$, and a Lagrange multiplier function $\mu \in \mathcal{F}(X)$ that satisfy the following.*

1. *Given a value function $V \in \mathcal{F}(X)$,*

$$(\tilde{c}_x, h_x, v_x, b'_x, q_x, \mu_x) \in \underset{\hat{c},\hat{h},\hat{v},\hat{b},\hat{q},\hat{\mu}}{\arg\max} \Big[u(\hat{c}) + \beta \mathbb{E}_s V(\hat{b}, s') \Big]$$

subject to

$$\hat{c} + \frac{\hat{b}}{R} \leq zF(1, \hat{h}, \hat{v}) - p_v \hat{v} - g(\hat{h}) + b,$$
$$-\frac{\hat{b}}{R} + \theta p_v \hat{v} \leq \kappa \hat{q},$$
$$\hat{q} u'(\hat{c}) = \beta \mathbb{E}_s \left[u'(\tilde{c}_{x'}) \Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'} \Big) + \mu_{x'} \kappa' q_{x'} \right],$$
$$0 = \hat{\mu} \left(\kappa \hat{q} + \frac{\hat{b}}{R} - \theta p_v \hat{v} \right), \qquad \hat{\mu} \geq 0,$$

with $x' = (\hat{b}, (z', R', \kappa'))$, for all $x = (b, s) = (b, (z, R, \kappa)) \in X$.

2. *The value function satisfies $V(b, s) = u(\tilde{c}_x) + \beta \mathbb{E}_s V(b'_x, s')$ for all $x = (b, s) \in X$.*

The two conditions in Definition 3 jointly define an MPE in $\{\tilde{c}, h, v, b', q, \mu\} \subset \mathcal{F}(X)$ with the associated value $V \in \mathcal{F}(X)$. The first condition postulates that it is suboptimal to deviate from $\{\tilde{c}, h, v, b', q, \mu\}$—the functions used to evaluate the conditional expectation in the asset pricing constraint (14)—in any state $x \in X$. The second condition postulates that the value function V solves the Bellman equation.

A brief inspection of the planner's constraints suggests that a time-consistent SP allocation need not be unique. When the collateral constraint is binding, any $\hat{\mu} \geq 0$ is a feasible choice for the current planner because $\hat{\mu}$ affects only the agent's complementary slackness conditions. Since $\hat{\mu}$ does not affect the planner's payoff, the planner is indifferent between any $\hat{\mu} \geq 0$. However, in an MPE, $\hat{\mu}$ must be consistent with the next-period planner's decision rule μ. The multiplicity of $\hat{\mu}$ thus generates a multiplicity of μ. In turn, μ affects the MPE asset price function q, the value of collateral, and thus the allocation functions $\{\tilde{c}, h, v, b'\}$. Consequently, there generally exist multiple welfare-ranked MPE.

Remark 3 (MPE multiplicity)**.** *There generally exist multiple welfare-ranked constrained-efficient MPE of Definition 3.*

Bianchi and Mendoza (2018) studied an MPE of Definition 3 under the assumption that the Lagrange multiplier function μ is restricted to satisfy the DE input optimality condition (12). If $\theta > 0$, (12) selects a *specific* MPE by leaving the social planner with a single degree of freedom relative to the DE outcome—namely, improving the allocation of bond holdings.[6] We refer the reader to their in-depth analysis of the resulting constrained-efficient allocation.

As we show next, however, there may exist a constrained-efficient MPE that entails the *unconstrained* allocation. This MPE welfare dominates any other constrained-efficient MPE.

3.3 Unconstrained allocation as a constrained-efficient plan

In this subsection, we identify the necessary conditions for the existence of a constrained-efficient MPE that entails the unconstrained allocation and characterize a candidate MPE with this property. Obtaining the sufficient conditions for the existence of such an MPE turns out difficult. However, after dropping the Markov perfection requirement, we derive the necessary and sufficient conditions for the existence of a constrained-efficient plan that entails the unconstrained allocation. Although such a plan is not Markovian, it is time consistent because there is no incentive to deviate from this plan.

Let $\left\{\tilde{c}^{\mathrm{UE}}, h^{\mathrm{UE}}, v^{\mathrm{UE}}, b^{\mathrm{UE}}\right\} \subset \mathcal{F}(X)$ denote the unconstrained allocation functions of Definition 2. According to Remark 2, these functions are part of the "unconstrained equilibrium" (**UE**)—the DE in the economy in which the agent's borrowing is not restricted by the collateral constraint. We are interested in a constrained-efficient MPE that entails the UE allocation functions.

Definition 4 (Unconstrained MPE)**.** *A constrained-efficient MPE of Definition 3 is an "unconstrained MPE" if* $\{\tilde{c}, h, v, b'\} = \left\{\tilde{c}^{UE}, h^{UE}, v^{UE}, b^{UE}\right\}$.

It will be useful to define the UE asset price function $q^{\mathrm{UE}} \in \mathcal{F}(X)$. We obtain it by evaluating (14) at the UE allocation and imposing $\mu_x = 0$ for all $x \in X$:

$$
\begin{aligned}
q_x^{\mathrm{UE}} &= \beta \mathbb{E}_s \left[\frac{u'\left(\tilde{c}_{x'}^{\mathrm{UE}}\right)}{u'\left(\tilde{c}_x^{\mathrm{UE}}\right)} \left(z' F_k \left(1, h_{x'}^{\mathrm{UE}}, v_{x'}^{\mathrm{UE}}\right) + q_{x'}^{\mathrm{UE}} \right) \right] \\
&= \sum_{t=1}^{\infty} \beta^t \sum_{s^t \in S^t} \Pr\left(s^t \mid s\right) \frac{u'\left(\tilde{c}_{x(s^t)}^{\mathrm{UE}}\right)}{u'\left(\tilde{c}_x^{\mathrm{UE}}\right)} z_t F_k \left(1, h_{x(s^t)}^{\mathrm{UE}}, v_{x(s^t)}^{\mathrm{UE}}\right), \qquad (18)
\end{aligned}
$$

where $x(s^t) = \left(b^{\mathrm{UE}}(x(s^{t-1})), s_t\right)$ for all $t \geq 1$ and $s^t \in S^t$, with $x(s^0) = x = (b, s) \in X$. Hence, q^{UE} is the present discounted value of dividends evaluated at the UE allocation.

[6] In the analysis of Bianchi and Mendoza (2018), the planner is, moreover, subject to the DE labor optimality condition (11). They show in Appendix A.1 (Proposition II) that (11), (12), and (15) are slack constraints in the sense that (11) is satisfied at the SP allocation and, if $\theta > 0$, (12) can be used to construct μ that satisfies (15) at the SP allocation. This construction selects a specific MPE of Definition 3, since, for a given allocation, many functions μ satisfy (15), but only one of them satisfies (12) if $\theta > 0$.

Let us also define the set of states $A \subset X$ at which the collateral constraint would be violated if it were imposed in the UE:

$$A = \left\{ x \in X \mid q_x^{\mathrm{UE}} < q_x^A \right\}, \tag{19}$$

where $q^A \in \mathcal{F}(X)$ is the asset price at which the collateral constraint would bind:

$$q_x^A = \frac{1}{\kappa} \left(-\frac{b_x^{\mathrm{UE}}}{R} + \theta p_v v_x^{\mathrm{UE}} \right). \tag{20}$$

Let $A^c = X \setminus A$ denote the complement of A. If $A = \emptyset$ (equivalently, $A^c = X$), the collateral constraint is irrelevant in the DE, and the DE allocation coincides with the UE allocation (Remark 2). Of course, the interesting (and empirically relevant) case is $A \neq \emptyset$.

Let $\mathbf{1}_Y$ denote the indicator function of a generic set Y. If the social planner could control the asset price directly, it would be sufficient to set $q_x = \mathbf{1}_A(x) q_x^A + \mathbf{1}_{A^c}(x) q_x^{\mathrm{UE}}$ to satisfy the collateral constraint for all $x \in X$ at the UE allocation. (One could interpret the planner's problem in Definition 2 as precisely this planning arrangement.) A constrained-efficient social planner of Definition 3, however, does not have such powers, and must abide by the asset pricing constraint (14). Inspecting the latter, we obtain the following necessary conditions for the existence of an unconstrained MPE.

Proposition 2 (Unconstrained MPE existence)**.** *There exists an unconstrained MPE of Definition 4 only if* $q_x \geq \max\left\{q_x^{UE}, q_x^A\right\}$ *and* $\mu_x \left(q_x - q_x^A\right) = 0$ *for all* $x \in X$*, and for all* $x = (b, s) \in A$*, there exist* $t \geq 1$ *and* $s^t \in S^t$ *such that* $x(s^t) \in \operatorname{cl} A$*, with* $x(s^t)$ *defined as in* (18)*.*

Proof. See Appendix A.2. ■

The first set of necessary conditions restricts an MPE asset price q in a rather intuitive way. The additional collateral value component implies that q is greater than or equal to the UE asset price q^{UE}. Moreover, since the collateral constraint evaluated at the UE allocation binds when the asset price is q^A, the MPE asset price q cannot be lower than the former. In fact, the MPE collateral constraint is equivalent to the condition $q_x \geq q_x^A$ for all $x \in X$. The final necessary condition in Proposition 2 is more subtle and requires the following: if the MPE visits the set A in which the collateral constraint is violated in the UE, then the closure of A must be visited at some point in the future. If $x \in A$, the UE asset price is too low. Since the UE allocation is given, the MPE asset price can be high enough only if the agent's Lagrange multipliers in the *next* periods are high enough. Hence, the collateral constraint must bind at some point in the future, which can happen only if the constraint will bind or will be violated in the UE in the next periods.

If A is in the ergodic set defined by the UE allocation, the last necessary condition in Proposition 2 will be satisfied. We assume that there exists a unique ergodic set.

Assumption 1 (Unconstrained ergodic set)**.** *Define the transition function* $P : X \times \mathcal{B}(X) \to [0,1]$ *as* $P((b,s), \hat{B} \times \hat{S}) = \mathbf{1}_{\hat{B}}\left(b^{UE}(b,s)\right) \Pr\Big(s' \in \hat{S} \mid s\Big)$*, where* $\mathcal{B}(X)$ *is the Borel* σ*-algebra on* X*. Then* P *generates a unique invariant probability measure* $\lambda : \mathcal{B}(X) \to [0,1]$ *that satisfies* $\lambda(\hat{X}) = \int_X P(x, \hat{X})\lambda(\mathrm{d}x)$ *for all* $\hat{X} \in \mathcal{B}(X)$*, with the associated ergodic set* $X^{UE} \subset X$*.*

As discussed previously, as long as $\beta\overline{R} < 1$, Assumption 1 will be satisfied and can be verified numerically. If $\lambda(A) = 0$, the collateral constraint is irrelevant almost everywhere (a.e.) on the UE ergodic set X^{UE}, and there exists an unconstrained MPE a.e. on X^{UE} with $q = q^{\mathrm{UE}}$ and $\mu_x = 0$ for all $x \in X^{\mathrm{UE}}$. A more interesting case is $\lambda(A) > 0$, so that the constraint would be violated in the UE on an infinite subset of the ergodic set. The next proposition suggests that one can generally construct a candidate MPE that satisfies almost all conditions of an unconstrained MPE.

Proposition 3 (Unconstrained MPE candidate)**.** *Suppose Assumption 1 holds and* $\lambda(A) > 0$*. There generally exist* $\{q, \mu\} \subset \mathcal{F}(X)$ *such that* $\left\{\tilde{c}^{UE}, h^{UE}, v^{UE}, b^{UE}, q, \mu\right\}$ *satisfy all conditions of Definition 3, except (14), possibly, holds approximately on A. If* $\lambda(A) = 1$*, then* $q = q^A$ *and* $\mu_x = \max\{\mu^q_x, 0\}$ *for all* $x \in X$*, where* $\mu^q \in \mathcal{F}(X)$ *is defined such that it satisfies*

$$q^A_x u'\left(\tilde{c}^{UE}_x\right) = \beta \mathbb{E}_s \left[u'\left(\tilde{c}^{UE}_{x'}\right)\left(z' F_k\left(1, h^{UE}_{x'}, v^{UE}_{x'}\right) + q^A_{x'}\right) + \mu^q_{x'} \kappa' q^A_{x'}\right], \tag{21}$$

with $x' = \left(b^{UE}_x, (z', R', \kappa')\right)$*, for all* $x = (b,s) \in X^{UE}$*, and* $\mu^q(\cdot) = 0$ *otherwise. If* $\mu^q_x \geq 0$ *on* X^{UE}*, then* $\left\{\tilde{c}^{UE}, h^{UE}, v^{UE}, b^{UE}, q, \mu\right\}$ *is an unconstrained MPE of Definition 4 a.e. on* X^{UE}*.*

Proof. See Appendix A.3. ■

The proof of Proposition 3 essentially describes a numerical strategy for constructing an unconstrained MPE of Definition 4. If $\lambda(A) \in (0,1)$, the construction involves finding a fixed point in the asset price function $q \in \mathcal{F}(X)$ and, at each iteration and for all $x \in A$, solving a system of linear equations in next-period Lagrange multipliers, defining $\mu \in \mathcal{F}(X)$. The solution to the linear system may or may not involve nonnegative multipliers. If it always does, the construction results in an unconstrained MPE.

A significant simplification is achieved if $\lambda(A) = 1$, so that the collateral constraint is violated in the UE a.e. on X^{UE}. We find this to be true numerically under the baseline calibration. In this case, we don't need to find a fixed point in the asset price, rather we can simply set $q = q^A$, which ensures that the collateral constraint holds with equality at the UE allocation. We do need to solve for the multipliers such that (21) holds on X^{UE}. A necessary condition for the multipliers to be nonnegative follows from (21).

Corollary 1. *Suppose Assumption 1 holds and $\lambda(A) = 1$. There exists an unconstrained MPE of Definition 4 a.e. on X^{UE} only if*

$$q_x^A u'\left(\tilde{c}_x^{UE}\right) \geq \beta \mathbb{E}_s \left[u'\left(\tilde{c}_{x'}^{UE}\right)\left(z' F_k\left(1, h_{x'}^{UE}, v_{x'}^{UE}\right) + q_{x'}^A\right)\right], \tag{22}$$

with $x' = \left(b_x^{UE}, (z', R', \kappa')\right)$, for all $x = (b, s) \in X^{UE}$.

The condition (22) can be verified numerically based on the UE allocation. This condition is necessary but not sufficient because for each $x \in A$, there may be several states $\hat{x} \in A$ that imply identical next-period bond holdings, i.e., $b^{\text{UE}}(\hat{x}) = b^{\text{UE}}(x)$. Solving the system of equations composed of (21) for each $\hat{x} \in \left\{\hat{x} \in A \mid b^{\text{UE}}(\hat{x}) = b^{\text{UE}}(x)\right\}$ for $\{\mu_{x'}^q\}$ may result in $\mu^q\left(b_x^{\text{UE}}, s'\right)$ being negative for some $s' \in S$. The difficulty in providing *sufficient* conditions for the existence of an unconstrained MPE is related to the essence of the Markov perfection requirement that forces the planner's decision rules to be functions of the payoff-relevant state variables only. The value of the Lagrange multiplier at a state $\left(b_x^{\text{UE}}, s'\right) \in X$ must be the same independently of the possible previous states $\left\{\hat{x} \in A \mid b^{\text{UE}}(\hat{x}) = b^{\text{UE}}(x)\right\}$.

If we drop the Markov perfection requirement, under Assumption 1 and $\lambda(A) = 1$, the condition (22) is both necessary and sufficient for the existence of a time-consistent constrained-efficient plan that entails the UE allocation. Recall that the classical definition of time consistency due to Strotz (1955), Kydland and Prescott (1977), and Calvo (1978) is the following: a plan $\{\{\pi_t(s^t \mid s^0)\}_{s^t \in S^t}\}_{t=0}^{\infty}$ is time consistent if for all $\tau > 0$ and $s^\tau \in S^\tau$, an optimal plan chosen at s^τ, $\{\{\pi_t(s^t \mid s^\tau)\}_{s^t \in S^t \mid s^\tau}\}_{t=\tau}^{\infty}$, satisfies $\pi_t(s^t \mid s^0) = \pi_t(s^t \mid s^\tau)$ for all $t \geq \tau$ and $s^t \in S^t \mid s^\tau$, where $S^t \mid s^\tau$ is the set of all histories s^t that continue from s^τ. Hence, any future reoptimization results in following the original plan chosen at $t = 0$. Such a plan is an outcome of a *subgame perfect equilibrium* of a game played by successive planners, but unlike in the case of an MPE, planners' strategies are not restricted to be Markovian. There is no incentive to deviate from a plan that entails the UE allocation, so any such plan is going to be time consistent according to the definition above.

Consider a plan chosen by a benevolent social planner that makes decisions once and for all at $t = 0$ subject to the sequential versions of the constraints in Definition 3.

Definition 5 (Constrained-efficient plan)**.** *An allocation $\{(\tilde{c}_t, h_t, v_t, b_{t+1})\}_{t=0}^{\infty}$, prices $\{q_t\}_{t=0}^{\infty}$,* *and Lagrange multipliers* $\{\mu_t\}_{t=0}^{\infty}$ *are a constrained-efficient plan if they solve*

$$\max_{\{(\tilde{c}_t, h_t, v_t, b_{t+1}, q_t, \mu_t)\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t)$$

subject to

$$\tilde{c}_t + \frac{b_{t+1}}{R_t} \leq z_t F(1, h_t, v_t) - p_v v_t - g(h_t) + b_t, \tag{23}$$

$$-\frac{b_{t+1}}{R_t} + \theta p_v v_t \leq \kappa_t q_t, \tag{24}$$

$$q_t u'(\tilde{c}_t) = \beta \mathbb{E}_t \left[u'(\tilde{c}_{t+1}) \Big(z_{t+1} F_k(1, h_{t+1}, v_{t+1}) + q_{t+1} \Big) + \mu_{t+1} \kappa_{t+1} q_{t+1} \right], \tag{25}$$

$$0 = \mu_t \left(\kappa_t q_t + \frac{b_{t+1}}{R_t} - \theta p_v v_t \right), \qquad \mu_t \geq 0, \tag{26}$$

for all $t \geq 0$ *and* $s^t \in S^t$*, given* $(b_0, s_0) = (b_0, (z_0, R_0, \kappa_0)) \in X$*.*

Let $\left\{ \left(\tilde{c}_t^{\mathrm{UE}}, h_t^{\mathrm{UE}}, v_t^{\mathrm{UE}}, b_{t+1}^{\mathrm{UE}} \right) \right\}_{t=0}^{\infty}$ be generated by the UE allocation functions of Definition 2 and $\left\{ q_t^A \right\}_{t=0}^{\infty}$ by q^A defined in (20). We then have the following proposition.

Proposition 4 (Unconstrained allocation as constrained-efficient plan)**.** *Suppose Assumption 1 holds,* $\lambda(A) = 1$*, and* (22) *holds for all* $x \in X^{UE}$*. Let* $x_0 = (b_0, s_0) \in X^{UE}$*. Then any constrained-efficient plan of Definition 5 entails the UE allocation* $\left\{ \left(\tilde{c}_t^{UE}, h_t^{UE}, v_t^{UE}, b_{t+1}^{UE} \right) \right\}_{t=0}^{\infty}$ *and is time consistent. An optimal* $\{(q_t, \mu_t)\}_{t=0}^{\infty}$ *is* $\{q_t\}_{t=0}^{\infty} = \left\{ q_t^A \right\}_{t=0}^{\infty}$*,* $\mu_0 = 0$*, and*

$$\mu_{t+1}((s^t, s_{t+1})) = \frac{q_t^A(s^t) u'\left(\tilde{c}_t^{UE}(s^t)\right) - \beta \mathbb{E}_t \left[u'\left(\tilde{c}_{t+1}^{UE}\right) \left(z_{t+1} F_k\left(1, h_{t+1}^{UE}, v_{t+1}^{UE}\right) + q_{t+1}^A \right) \right]}{\beta \mathbb{E}_t \left(\kappa_{t+1} q_{t+1}^A \right)}, \tag{27}$$

for all $t \geq 0$*,* $s^t \in S^t$*, and* $s_{t+1} \in S$*.*

Proof. See Appendix A.4. ■

Proposition 4 is a consequence of Proposition 3 and Corollary 1. A sequential planning problem of Definition 5 involves (25) for each history $s^t \in S^t$, and the simplest way to satisfy the former given the UE allocation and $\{q_t\}_{t=0}^{\infty} = \left\{ q_t^A \right\}_{t=0}^{\infty}$ is making $\mu_{t+1}((s^t, s_{t+1}))$ constant over $s_{t+1} \in S$, which gives (27). Given (22), the multipliers are guaranteed to be nonnegative. Note that $\{\mu_t\}_{t=0}^{\infty}$ constructed according to (27) is not Markovian since $\mu_{t+1}(s^{t+1})$ depends on the allocation and prices at s^t.

The possible existence of constrained-efficient equilibria that entail the UE allocation has important implications for the design of optimal policy that we now explore.

4 Optimal Policy

In this section, we characterize the policies that can implement constrained-efficient equilibria as regulated competitive equilibria. As a policy tool, we focus on a tax on debt rebated lump sum—the instrument suggested in the existing literature (Bianchi, 2011; Bianchi and Mendoza, 2018; Jeanne and Korinek, 2019).

Although a positive debt tax generally discourages borrowing for given asset prices, the tax tends to decrease asset prices, making the collateral constraint more likely to bind for a given level of debt, and thus has an ambiguous effect on welfare and the probability of financial crises. As a consequence of this tradeoff, in the presence of the working capital constraint ($\theta > 0$), the optimal time-consistent policy generally requires both taxing debt when the collateral constraint is slack (good times) and *subsidizing* debt when the collateral constraint binds (bad times). In this case, the optimal policy induces *a* constrained-efficient MPE, but cannot achieve an unconstrained MPE. Implementing the latter requires, in addition, subsidizing intermediate inputs.

In the absence of the working capital constraint ($\theta = 0$), the tax on debt can induce an unconstrained MPE. Under the premise of Proposition 4, the optimal time-consistent policy requires only subsidizing (but not taxing) debt. Therefore, there is an important discontinuity in the optimal time-consistent policy at $\theta = 0$.

4.1 Regulated competitive equilibrium

Suppose the government imposes a tax on debt τ_t and provides a lump-sum transfer T_t. The agent's budget constraint (2) becomes

$$c_t + q_t k_{t+1} + \frac{b_{t+1}}{R_t(1+\tau_t)} \leq z_t F(k_t, h_t, v_t) - p_v v_t + q_t k_t + b_t + T_t.$$

A positive τ_t increases the effective interest rate, subsidizing saving and taxing borrowing. Conversely, a negative τ_t subsidizes borrowing. The government budget constraint is $T_t = \left(\frac{1}{1+\tau_t} - 1\right) \frac{b_{t+1}}{R_t}$, so the tax income is rebated back to the agent. Consequently, the role of τ_t is solely to affect the agent's borrowing decision but not to distort the allocation otherwise.

Given Definition 1 and Remark 1, we define a regulated competitive equilibrium (**CE**) in recursive form as follows.

Definition 6 (Regulated competitive equilibrium)**.** *Given a tax function $\tau \in \mathcal{F}(X)$, a regulated competitive equilibrium is a set of allocation functions $\{\tilde{c}, h, v, b'\} \subset \mathcal{F}(X)$, an asset price function $q \in \mathcal{F}(X)$, and a Lagrange multiplier function $\mu \in \mathcal{F}(X)$ that satisfy*

(9)–(12), (14), (15), *and*

$$u'(\tilde{c}_x) = (1+\tau_x)\Big(\beta R \mathbb{E}_s u'(\tilde{c}_{x'}) + \mu_x\Big) \qquad \text{if } b'_x \in \operatorname{int} B, \tag{28}$$

with $x' = (b'_x, (z', R', \kappa'))$, *for all* $x = (b, s) = (b, (z, R, \kappa)) \in X$.

Note that the only difference between the DE conditions in Remark 1 and the regulated CE conditions in Definition 6 is the bond Euler equation (28) that has replaced (13). Since $u'(\cdot) > 0$, the regulated CE exists only if $\tau_x > -1$ for all $x \in X$.

To develop the intuition on how this policy tool works, let us consider a *one-shot* change $\mathrm{d}\tau_x$ in the tax rate—that is, assuming that the initial tax function τ is used again starting from the next period. Let $X^\tau_u = \{x \in X \mid \mu_x = 0\}$ denote the subset of the state space in which the collateral constraint is weakly slack ("unconstrained region") and let $X^\tau_c = X \setminus X^\tau_u$ denote the complement of X^τ_u in X—the "constrained region" in which the collateral constraint is strictly binding. Both X^τ_u and X^τ_c are conditional on the initial tax function, as indicated by the superscript τ. The next proposition investigates the equilibrium adjustment in response to a marginal one-shot tax change when the collateral constraint is initially either *strictly* slack ($x \in \operatorname{int} X^\tau_u$) or strictly binding ($x \in X^\tau_c$)—hence, the equilibrium outcome stays in the same state-space region as before the tax change, and the differentials are well defined.

Proposition 5 (One-shot tax change)**.** *Consider a regulated CE of Definition 6 given* τ *and a one-shot tax change* $\mathrm{d}\tau_x$ *at the state* $x \in X$. *If* $x \in \operatorname{int} X^\tau_u$, *then* $\mathrm{d}h_x = \mathrm{d}v_x = \mathrm{d}\mu_x = 0$,

$$\mathrm{d}b'_x = \left[-\frac{u''(\tilde{c}_x)}{R}\frac{1}{1+\tau_x} - \beta R \mathbb{E}_s\left(u''(\tilde{c}_{x'})\frac{\partial \tilde{c}_{x'}}{\partial b'_x}\right)\right]^{-1} \frac{u'(\tilde{c}_x)}{(1+\tau_x)^2}\,\mathrm{d}\tau_x\,, \tag{29}$$

$$\mathrm{d}q_x = \frac{1}{u'(\tilde{c}_x)}\left\{\frac{1}{R}q_x u''(\tilde{c}_x) + \beta \mathbb{E}_s \frac{\partial}{\partial b'_x}\left[u'(\tilde{c}_{x'})\Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'}\Big) + \mu_{x'}\kappa' q_{x'}\right]\right\}\mathrm{d}b'_x\,, \tag{30}$$

and $\mathrm{d}\tilde{c}_x = -\frac{1}{R}\,\mathrm{d}b'_x$, *provided the partial derivatives that appear in* (29) *and* (30) *exist. If* $x \in X^\tau_c$ *and* $\theta = 0$, *then* $\mathrm{d}\tilde{c}_x = \mathrm{d}h_x = \mathrm{d}v_x = \mathrm{d}b'_x = \mathrm{d}q_x = 0$ *and* $\mathrm{d}\mu_x = -u'(\tilde{c}_x)(1+\tau_x)^{-2}\,\mathrm{d}\tau_x$. *If* $x \in X^\tau_c$ *and* $\theta > 0$, *the differentials are given by* (A.13)–(A.18).

Proof. See Appendix A.5. ■

If $x \in \operatorname{int} X^\tau_u$, the tax on debt has an interpretation of a *macroprudential* tax—that is, a tax applied in "good times." In this case, $\mathrm{d}\mu_x = \mu_x = 0$, so that the equilibrium labor h_x and intermediate inputs v_x are pinned down by (11) and (12) independently of τ and remain constant. Consequently, the right-hand side in the country budget constraint (9) also remains constant, and thus net consumption $\tilde{c}_x$ and next-period bond holdings b'_x adjust proportionally in the opposite directions. The response in b'_x, given by (29), is then determined by the bond Euler equation (28): in particular, it depends on how the current and future marginal utility of consumption adjust with b'_x. For a given future marginal utility, an increase in τ_x raises the current marginal utility, which requires a decrease in consumption and an increase in saving. A change in b'_x, however, affects the next-period state, and thus future consumption and marginal utility. These two effects are captured in the two terms involving u'' in (29). If the equilibrium net consumption function $\tilde{c}$ is increasing in bond holdings—a property that is true numerically in the DE under the baseline calibration—we have $\partial \tilde{c}_{x'}/\partial b'_x \geq 0$ in (29), which ensures that an increase in the tax rate raises saving. We state this fact in Corollary 2.

Corollary 2. *In the context of Proposition 5, if $\tilde{c}$ is increasing in b for all $s \in S$, then* $\mathrm{d}b'_x/\mathrm{d}\tau_x > 0$ *for all* $x \in \operatorname{int} X^\tau_u$.

The response in the asset price q_x, given by (30), is proportional to the response in b'_x. Like the latter, it has an intertemporal nature, depending on how the current and future determinants of q_x vary with b'_x in the asset pricing condition (14). Since the collateral constraint is strictly slack, an increase in b'_x decreases consumption and raises the current marginal utility, decreasing the asset price through the SDF—the term $\frac{1}{R} q_x u''(\tilde{c}_x) < 0$ in (30). At the same time, an increase in b'_x affects the expected next-period payoff on capital in marginal utility units—the second term in (30)—and the overall response in the asset price depends on the sign and magnitude of this second effect. We find that in the DE, numerically, a fall in the future marginal utility due to an increase in future consumption dominates the changes in the future marginal product of capital, asset price, and Lagrange multiplier, so the partial derivative $\partial/\partial b'_x$ in (30) is nonpositive, and the asset price falls in response to a macroprudential tax. Corollary 3 formalizes this statement.

Corollary 3. *In the context of Proposition 5, if $\tilde{c}$ is increasing and* $u'(\tilde{c}_x)(zF_k(1, h_x, v_x) + q_x) + \mu_x \kappa q_x$ *is decreasing in b for all $s \in S$, then* $\mathrm{d}q_x/\mathrm{d}\tau_x < 0$ *for all* $x \in \operatorname{int} X^\tau_u$.

Corollaries 2 and 3 demonstrate a trade-off associated with macroprudential policy. On the one hand, it discourages borrowing, making the collateral constraint less likely to bind. On the other hand, it depresses asset prices, making the collateral constraint more likely to bind. Hence, its welfare benefits are in general ambiguous.

If the collateral constraint is initially strictly binding ($x \in X^\tau_c$) and $\theta = 0$, the allocation and asset price are pinned down by (9), (11), (12), (14), and (15) independently of τ, so the change $\mathrm{d}\tau_x$ affects only the Lagrange multiplier μ_x determined from (28), and $\mathrm{d}\mu_x/\mathrm{d}\tau_x < 0$. If $\theta > 0$, the effects of a one-shot tax change are more complex. In this case, the adjustment in b'_x and q_x causes a reallocation in intermediate inputs v_x, consistent with the binding constraint in (15). In turn, the change in inputs leads to a change in labor h_x in (11), net consumption $\tilde{c}_x$ in (9), and the Lagrange multiplier μ_x in (12). These changes lead to further adjustments in b'_x and q_x in (14) and (28), and so on. Our numerical analysis shows that Corollaries 2 and 3 mostly continue to hold in the binding region as well.

4.2 Optimal time-consistent policy

We now characterize the optimal time-consistent tax on debt. We begin with a proposition that characterizes the optimal policy in relation to constrained-efficient equilibria of Definition 3. The main result is that if $\theta > 0$—the case also analyzed by Bianchi and Mendoza (2018)—the optimal policy implements a *specific* constrained-efficient MPE, but not an unconstrained (welfare-dominant) MPE. However, if $\theta = 0$, we show that the tax on debt can implement any constrained-efficient MPE, including an unconstrained MPE. We obtain a general expression for the optimal tax rate in terms of the corresponding constrained-efficient outcome (for any $\theta \geq 0$). We then provide several corollaries that explore the sign of the optimal tax rate depending on whether the collateral constraint is slack or binding and show that the optimal policy that implements an unconstrained MPE entails a debt *subsidy*. We also show that if $\theta > 0$, an unconstrained MPE can be implemented with a debt subsidy complemented by a subsidy to intermediate inputs.

Proposition 6 (Optimal time-consistent tax on debt)**.** *Consider an MPE of a game in which successive policymakers choose the tax function $\tau \in \mathcal{F}(X)$ to achieve the best regulated CE of Definition 6, taking the next-period decision rules as given. If $\theta = 0$, the set of such MPE is equivalent to the set of constrained-efficient MPE of Definition 3. If $\theta > 0$, the equilibrium τ induces one specific constrained-efficient MPE corresponding to $\mu \in \mathcal{F}(X)$ defined as*

$$\mu_x = \frac{u'(\tilde{c}_x)}{u'(\tilde{c}_x) - \mu_x^{SP} \kappa q_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)}} \mu_x^{SP}, \tag{31}$$

where μ_x^{SP} is the policymaker's shadow value of collateral. If $\lambda(A) > 0$, this MPE is not an unconstrained MPE of Definition 4.

The tax function that induces a constrained-efficient MPE $\{\tilde{c}, h, v, b', q, \mu\} \subset \mathcal{F}(X)$ as a regulated CE satisfies

$$\tau_x = \tau_x^{MP} + \tau_x^{EP}, \tag{32}$$

where

$$\tau_x^{MP} \equiv -\frac{\mathbb{E}_s \left(\mu_{x'}^{SP} \kappa' q_{x'} \frac{u''(\tilde{c}_{x'})}{u'(\tilde{c}_{x'})} \right)}{\mathbb{E}_s u'(\tilde{c}_{x'}) + \frac{\mu_x}{\beta R}} \geq 0 \tag{33}$$

and

$$\tau_x^{EP} \equiv \underbrace{\frac{\mu_x^{SP} - \mu_x}{\beta R \mathbb{E}_s u'(\tilde{c}_{x'}) + \mu_x}}_{\textit{risk sharing component}}$$

$$+ \underbrace{\mu_x^{SP} \frac{\kappa}{u'(\tilde{c}_x)} \frac{q_x u''(\tilde{c}_x) + \beta R \mathbb{E}_s \frac{\partial}{\partial b'_x}\left[u'(\tilde{c}_{x'})\Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'}\Big) + \mu_{x'} \kappa' q_{x'}\right]}{\beta R \mathbb{E}_s u'(\tilde{c}_{x'}) + \mu_x}}_{\textit{collateral externality component}}, \quad (34)$$

provided the partial derivative $\partial/\partial b'_x$ *that appears in* (34) *exists, and where* μ *must be consistent with* (31) *if* $\theta > 0$.

Proof. See Appendix A.6. ■

According to Proposition 6, in the absence of the working capital constraint ($\theta = 0$), the tax on debt can implement any constrained-efficient MPE of Definition 3, including an unconstrained MPE of Definition 4 if it exists. In this case, the planner can vary the agent's shadow value of collateral μ when the collateral constraint binds, affecting the equilibrium asset price, and expanding the set of feasible allocations. (See the discussion preceding Remark 3.) If $\theta > 0$, however, the tax on debt can implement only one specific constrained-efficient MPE, and this MPE is not an unconstrained MPE, provided that the collateral constraint is relevant on the UE ergodic set (i.e., $\lambda(A) > 0$). This is because the working capital constraint restricts the Lagrange multiplier function μ to be consistent with the DE optimal input condition (12), introducing a wedge relative to the UE condition (16) and generating a one-to-one mapping between the planner's and agent's shadow values of collateral given by (31). Note that $u'(\tilde{c}_x)$ and $u'(\tilde{c}_x) - \mu_x^{\mathrm{SP}} \kappa q_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)}$ are the agent's and planner's shadow values of income (Lagrange multipliers on the country budget constraint), so (31) requires the equality between the agent's and planner's shadow values of collateral normalized by the corresponding shadow values of income.

It is important to emphasize that there is a single-valued mapping between a constrained-efficient MPE and a tax function that induces that MPE as a regulated CE. The tax is uniquely determined not only in the states in which the collateral constraint is slack but also in the binding states, unlike in the analysis of Bianchi (2011), Schmitt-Grohé and Uribe (2017), and Jeanne and Korinek (2019), who all find that the tax on debt cannot affect the allocation in the binding state—it can only affect the agent's shadow value of collateral. In our environment, if $\theta > 0$, the agent's shadow value of collateral μ is linked to the input allocation through (12), so when the collateral constraint binds, there is still room for optimal reallocation between bond holdings and inputs in (15) by adjusting the tax rate. If $\theta = 0$, *for given next-period decision rules*, the current tax rate τ_x maps to the current multiplier μ_x in (28) in the binding state, and the allocation and asset price are pinned down by the remaining constraints in the regulated CE of Definition 6. However, in an MPE, the current μ_x must be consistent with the function μ that affects the next-period payoff on capital in (14). Hence, in a given MPE, the value of μ_x is given, and (28) provides a unique value of τ_x that maps to the former. This property of the optimal time-consistent policy arises because the shadow value of collateral affects the equilibrium asset price, while there is no such effect in the alternative environments mentioned above.

The tax function that implements a given constrained-efficient MPE is described by (32)–(34), comprising two components. The first component is a "macroprudential component" τ^{MP} given by (33). This component reflects that greater current savings b'_x relax the next-period country budget constraint, having a positive effect on future net consumption $\tilde{c}_{x'}$ and asset price $q_{x'}$, and thus relaxing the collateral constraint in the next-period states $x' = (b'_x, s')$ in which the constraint is strictly binding ($\mu^{\text{SP}}_{x'} > 0$). The macroprudential component is nonnegative and captures the planner's motive to subsidize savings (tax debt issuance) today to prevent or mitigate financial crises in the future. If the collateral constraint is slack in the current period, the agent's and planner's complementary slackness conditions require $\mu_x = \mu^{\text{SP}}_x = 0$ in any MPE. In this case, the macroprudential component is the only component of the optimal tax.

Corollary 4. *Consider the optimal time-consistent tax on debt described in* (32)–(34)*. If the collateral constraint is slack at x, then*

$$\tau_x = \tau^{MP}_x = -\frac{\mathbb{E}_s \left(\mu^{SP}_{x'} \kappa' q_{x'} \frac{u''(\tilde{c}_{x'})}{u'(\tilde{c}_{x'})} \right)}{\mathbb{E}_s u'(\tilde{c}_{x'})}. \tag{35}$$

Given (A.22)*, τ^{MP} in* (35) *is equivalent to the "macroprudential debt tax" in Bianchi and Mendoza (2018, eq. (17), p. 605).*

The second component of the optimal time-consistent tax on debt is an "ex post component" τ^{EP} given by (34). By Corollary 4, the ex post component is active only if the collateral constraint is binding. The ex post component is, in turn, a sum of two terms: a "risk sharing component" and a "collateral externality component."

The risk sharing component is proportional to the difference between the planner's and agent's shadow values of collateral ($\mu^{\text{SP}}_x - \mu_x$). If $\mu^{\text{SP}}_x > \mu_x$, the representative agent undervalues the utility benefit of increased consumption smoothing achieved by relaxing the binding collateral constraint, and the planner has an incentive to subsidize savings in the binding state. Conversely, if $\mu^{\text{SP}}_x < \mu_x$, the representative agent overvalues savings in the binding state, and the negative risk sharing component provides an incentive to tax savings (subsidize debt) when the collateral constraint binds. If $\theta > 0$, (31) implies $\mu_x^{\text{SP}} > \mu_x$ whenever $\mu_x^{\text{SP}} > 0$.

Corollary 5. *Consider the optimal time-consistent tax on debt described in (32)–(34). If μ is constrained by (31), the risk sharing component of τ^{EP} is positive in the states in which the collateral constraint is strictly binding.*

The collateral externality component arises due to the pecuniary externality (Dávila and Korinek, 2018) and reflects the effect of greater current savings b'_x on the current asset price q_x. First, greater current savings crowd out current net consumption $\tilde{c}_x$ and have a negative effect on the asset price through the current marginal utility of consumption, which corresponds to $q_x u''(\tilde{c}_x) < 0$ in the numerator in (34). Second, a change in b'_x affects the next-period state $x' = (b'_x, s')$, and thus the next-period payoff on capital in marginal utility units, i.e., $u'(\tilde{c}_{x'})\Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'}\Big) + \mu_{x'}\kappa' q_{x'}$. The latter effect depends on the monotonicity properties of the MPE decision rules.

Corollary 6. *Consider the optimal time-consistent tax on debt described in (32)–(34). If the MPE payoff on capital in marginal utility units is decreasing in b for all $s \in S$, the collateral externality component of τ^{EP} is negative in the states in which the collateral constraint is strictly binding.*

The sufficient condition for a negative collateral externality in Corollary 6 is similar to the sufficient condition for a negative effect of a one-shot tax change on the the asset price in Corollary 3. It is satisfied if an increase in future consumption (a decrease in the marginal utility of consumption) dominates the effects through the changes in the future marginal product of capital, asset price, and agent's shadow value of collateral. If an increase in current savings does decrease the current asset price, so that the collateral externality component is negative, the planner has an incentive to tax savings (subsidize debt issuance) in order to increase the current asset price and relax the binding collateral constraint.

A striking simplification of the optimal time-consistent tax on debt arises if it implements an unconstrained MPE of Definition 4. In this case, the policymaker acts as if being subject to the country budget constraint only, consistent with the unconstrained problem of Definition 2, while $\{q, \mu\}$ are set to satisfy the collateral constraint, asset pricing constraint, and the agent's complementary slackness conditions at the UE allocation, as described in Proposition 3. Consequently, the collateral constraint can be dropped from the planner's best response problem, which implies $\mu_x^{\text{SP}} = 0$ for all $x \in X$. Imposing the latter in (33) and (34), we observe that both the macroprudential and collateral externality components become zero, and the optimal tax is given by the nonpositive risk sharing component.

Corollary 7. *The optimal time-consistent tax on debt that implements an unconstrained MPE of Definition 4 is given by*

$$\tau_x = \tau_x^{EP} = \frac{-\mu_x}{u'\left(\tilde{c}_x^{UE}\right) + \mu_x} \in (-1, 0], \tag{36}$$

where the denominator has been simplified with (17).

The optimal time-consistent policy given by (36) is a debt *subsidy* that responds to the agent's shadow value of collateral μ, closing the wedge in the DE Euler equation (13) introduced by the Lagrange multiplier. The optimal policy thus ensures that the representative agent can borrow as much as it is optimal to do at the UE allocation, counteracting the distortion introduced by the collateral constraint.

By Proposition 6, the tax on debt can implement an unconstrained MPE only if $\theta = 0$. If $\theta > 0$, a wedge in the input optimality condition (12) prevents the implementation of an unconstrained MPE. Not surprisingly, if the planner can close that wedge by subsidizing inputs, an unconstrained MPE, if it exists, can be implemented.

Proposition 7 (Optimal time-consistent policy)**.** *Suppose, in addition to a tax on debt, a policymaker can tax intermediate inputs, so that the agent's budget constraint (2) becomes*

$$c_t + q_t k_{t+1} + \frac{b_{t+1}}{R_t(1+\tau_t)} \leq z_t F(k_t, h_t, v_t) - (1+\tau_t^v) p_v v_t + q_t k_t + b_t + T_t,$$

and the government budget constraint is $T_t = \left(\frac{1}{1+\tau_t} - 1\right) \frac{b_{t+1}}{R_t} + \tau_t^v p_v v_t$*. Then the set of MPE of a game in which successive policymakers choose* $\{\tau, \tau^v\} \subset \mathcal{F}(X)$ *to achieve the best regulated CE, taking the next-period decision rules as given, is equivalent to the set of constrained-efficient MPE of Definition 3. The policy that induces a given constrained-efficient MPE as a regulated CE satisfies (32)–(34) and*

$$\tau_x^v = \theta \left(\frac{\mu_x^{SP}}{u'(\tilde{c}_x) - \mu_x^{SP} \kappa q_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)}} - \frac{\mu_x}{u'(\tilde{c}_x)} \right). \tag{37}$$

The policy that implements an unconstrained MPE of Definition 4 is given by (36) and

$$\tau_x^v = -\frac{\theta \mu_x}{u'\left(\tilde{c}_x^{UE}\right)} \leq 0. \tag{38}$$

Proof. See Appendix A.7. ■

By Proposition 7, taxes on debt and inputs jointly can implement the whole set of constrained-efficient MPE of Definition 3. Consistent with Proposition 6, (37) implies that the tax on inputs is required only if $\theta > 0$ and only in the states in which the collateral constraint binds. In this case, the tax is proportional to the difference between the planner's and agent's shadow values of collateral normalized by the corresponding shadow values of income. If the representative agent underestimates the normalized shadow value of collateral, it is optimal to levy a positive input tax. Conversely, if the agent's normalized shadow value of collateral is greater than the planner's, it is optimal to subsidize inputs in the binding state. If the normalized shadow values are exactly equal, the optimal input tax is zero, and the tax on debt induces the specific MPE that satisfies (31).

Note that the optimal tax on debt in Proposition 7 is characterized by the same conditions (32)–(34) as in Proposition 6. In particular, Corollaries 4–7 continue to hold. However, if $\theta > 0$, to implement an unconstrained MPE of Definition 4, we need to augment the debt subsidy given by (36) with an input *subsidy* given by (38) that equalizes the effective marginal cost of inputs with its price, consistent with (16).

The tax rates in (36) and (38) have simple expressions, but they involve the agent's shadow value of collateral μ that is part of an unconstrained MPE. Propositions 2 and 3 and Corollary 1 could go as far as providing the necessary conditions for the existence of an unconstrained MPE and describing a candidate unconstrained MPE, lacking the sufficient conditions. However, if we drop the Markov perfection requirement, under the premise of Proposition 4, (27) provides a closed-form expression for a history-contingent $\{\mu_t\}_{t=0}^{\infty}$ that is part of a time-consistent constrained-efficient plan of Definition 5 that entails the UE allocation. Using that expression together with (36) and (38), we can construct the corresponding time-consistent policy $\{(\tau_t, \tau_t^v)\}_{t=0}^{\infty}$ that implements the UE allocation in a regulated CE. We state this formally in Proposition 8.

Proposition 8 (Implementing unconstrained allocation)**.** *Under the premise of Proposition 4, define $\{\mu_t\}_{t=0}^{\infty}$ by (27) with $\mu_0 = 0$ and define a policy $\{(\tau_t, \tau_t^v)\}_{t=0}^{\infty}$ by (36) and (38) stated in sequential notation (i.e., replacing x with t). Then this policy implements the UE allocation $\left\{\left(\tilde{c}_t^{UE}, h_t^{UE}, v_t^{UE}, b_{t+1}^{UE}\right)\right\}_{t=0}^{\infty}$ in a regulated CE, is Ramsey-optimal and time consistent.*

Proof. See Appendix A.8. ■

5 Quantitative Results

In this section, we discuss the quantitative properties of the optimal policies. We show that, under the calibration of Bianchi and Mendoza (2018), the ex post component of the optimal time-consistent tax on debt is a *subsidy*, and its magnitude can be larger than that of the macroprudential component. In particular, we demonstrate that a significant subsidy must be provided during financial crisis events. We then show that both the ex post debt subsidies and ex ante macroprudential taxes are essential for achieving welfare gains from the time-consistent policy. If either component is restricted, a time-consistent policy can lead to welfare losses. Finally, we compute the policy that implements the welfare-dominant unconstrained allocation. We show that this policy entails debt and input subsidies (but not taxes), and the resulting allocation features significantly more debt than the DE.

5.1 Calibration and computation

We calibrate the model identically to Bianchi and Mendoza (2018), extracting the exact parameter values, Markov chain states, and the transition matrix from their replication package. The model period is a year. The agent's preferences and technology satisfy

$$u(\tilde{c}) = \lim_{\hat{\sigma}\to\sigma} \frac{\tilde{c}^{1-\hat{\sigma}} - 1}{1-\hat{\sigma}}, \qquad g(h) = \chi \frac{h^{1+\omega}}{1+\omega}, \qquad F(k,h,v) = k^{\alpha_k} h^{\alpha_h} v^{\alpha_v}, \tag{39}$$

where $\sigma, \chi > 0$, $\omega \geq 0$, $\alpha_k, \alpha_h, \alpha_v > 0$, and $\alpha_k + \alpha_h + \alpha_v \leq 1$. The structural parameter values are summarized in Table 1.[7]

Table 1: Parameter values

Parameter	Value	Description
		Preferences
β	0.95	Discount factor
σ	1.1	Inverse intertemporal elasticity of substitution
ω	0.5	Inverse Frisch elasticity of labor supply
χ	0.64	Labor disutility scale
		Technology
α_k	0.01	Capital share in production
α_h	0.352	Labor share in production
α_v	0.45	Input share in production
θ	0.163	Input share financed in advance
p_v	0.818	Input price

Notes: See Bianchi and Mendoza (2018, Section III.A) for calibration details. That section reports $\sigma = 1$ and $\chi = 0.352$. (We use the values from the replication package.)

Since $\theta > 0$, by Proposition 6, the tax on debt can implement a specific constrained-efficient MPE of Definition 3 consistent with (31). We are going to refer to this MPE (its allocation) as the SP equilibrium (SP allocation).

[7] Consistent with Bianchi and Mendoza (2018, footnote 16 on p. 611 and the replication package), we subtract constant government spending of 0.13 from the right-hand side of the country budget constraint (9). This lump-sum term does not affect the theoretical results.

The Markov chain $\{s_t\}$ has three TFP states $z_t \in \{1.78, 1.82, 1.86\}$, three gross interest rate states $R_t \in \{0.987, 1.012, 1.037\}$, and two credit regimes $\kappa_t \in \{\kappa^L, \kappa^H\} = \{0.75, 10\}$.[8] The components of s_t are mutually independent. The collateral constraint binds at the DE or SP allocations only if $\kappa = \kappa^L$. Moreover, $\Pr\big(\kappa_{t+1} = \kappa^L \mid \kappa_t = \kappa^L\big) = 0$, which matches the one-year duration of financial crises.

We develop an independent MATLAB code for our quantitative analysis. The details are provided in Appendix B. We compute all equilibria using global nonlinear methods, approximating the unknown functions with linear splines. We use the same grid for bond holdings as in Bianchi and Mendoza (2018) except when computing the UE allocation, which requires setting the lower bound $\underline{b}$ close to minus the natural borrowing limit. The DE we obtain matches closely the DE in Bianchi and Mendoza (2018), but there are certain differences between the SP allocations. In particular, we find less overborrowing by private agents when the collateral constraint is slack, underborrowing when the constraint binds, and greater welfare gains. These differences stem from the differences in addressing the nonconvexities of the planner's choice set.

When computing the SP equilibrium, we do not rely on any differentiability assumptions, although, by construction, the piecewise linear decision rules we obtain are differentiable a.e. To compute the optimal time-consistent tax on debt, we use the primal approach, backing out τ from (28). To decompose τ into its macroprudential and ex post components, we note that the calibration of $\{\kappa_t\}$ implies $\Pr\big(\mu^{\mathrm{SP}}_{x'} > 0 \mid \mu^{\mathrm{SP}}_x > 0\big) = 0$. Hence, (32)–(34) combined with Corollary 4 imply $\tau_x = \tau^{\mathrm{MP}}_x$ when $\mu_x = \mu^{\mathrm{SP}}_x = 0$ and $\tau_x = \tau^{\mathrm{EP}}_x$ when $\mu_x > 0$ (equivalently, $\mu^{\mathrm{SP}}_x > 0$ due to (31)), where μ is backed out from (12).

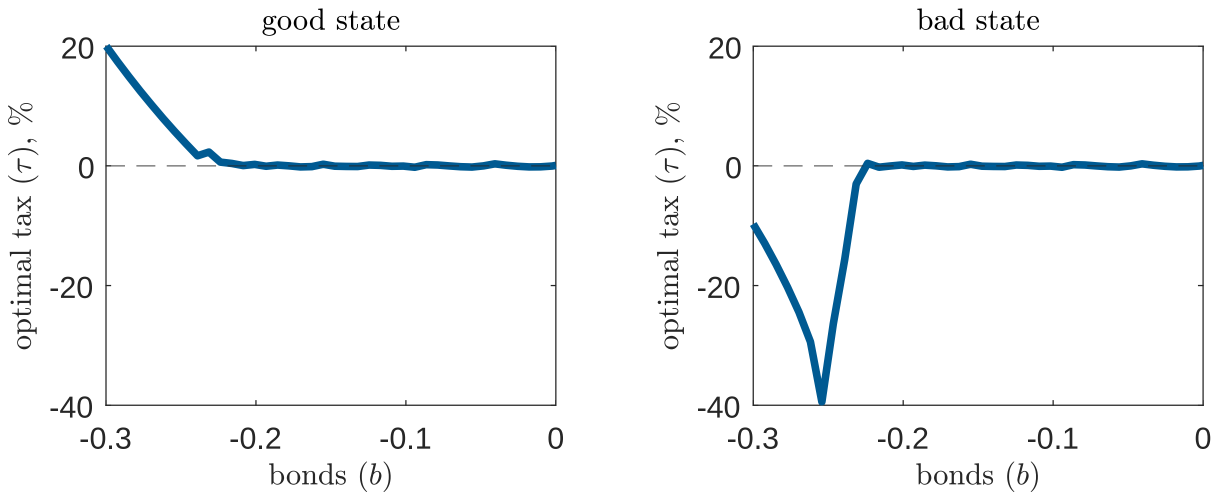
5.2 Optimal financial crisis management

Figure 1 plots the optimal time-consistent tax on debt τ of Proposition 6 as a function of bond holdings b on the horizontal axis in the two exogenous states s that both have average z and high R but differ in the value of κ: $\kappa = \kappa^H$ in the left panel ("good state") and $\kappa = \kappa^L$ in the right panel ("bad state").

In the good state, the collateral constraint is slack for all b (see Figure 3). By Corollary 4, $\tau_x = \tau^{\mathrm{MP}}_x \geq 0$. For $b \leq -0.2$, the constraint may bind in the next period with a positive probability, i.e., $\Pr\big(\mu^{\mathrm{SP}}_{x'} > 0\big) > 0$, in which case $\tau_x = \tau^{\mathrm{MP}}_x > 0$, consistent with (35). In this region, as b increases, next-period bond holdings b'_x slightly trend upwards (Figure 3), and the marginal propensity to consume out of greater asset income is close to 1. By (28), a

[8] Unlike the replication package, Section III.A in Bianchi and Mendoza (2018) reports $\kappa^H = 0.9$. We verified that any $\kappa^H \geq 0.91$ ($\kappa^H \geq 0.93$) generates the same DE (SP) allocation. Hence, κ^H can be set significantly lower than 10 without affecting the quantitative results.

Figure 1: Optimal tax in good and bad state



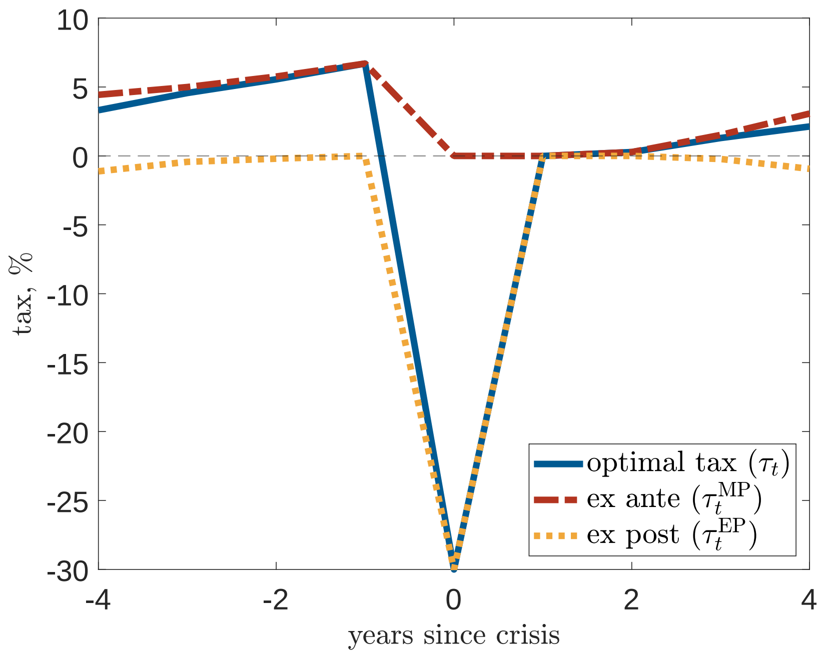
Notes: "good state" = (average z, high R, high κ), "bad state" = (average z, high R, low κ).

decrease in the marginal utility of consumption translates into a decrease in τ from around 20% to zero when $\Pr\left(\mu_{x'}^{\mathrm{SP}} > 0\right) = 0$. This is consistent with (35), since an increase in b'_x induces a fall in the planner's next-period shadow value of collateral $\mu_{x'}^{\mathrm{SP}}$ in the states in which the constraint binds, implying a smaller tax.

In the bad state, the collateral constraint binds for $b \leq -0.23$ (see Figure 4). In this region, the constraint is slack in the next period, which implies $\tau_x^{\mathrm{MP}} = 0$. We observe from Figure 1 that $\tau_x = \tau_x^{\mathrm{EP}} < 0$ when the constraint binds. This means that the (negative) collateral externality component in (34) dominates the (positive, by Corollary 5) risk sharing component of τ^{EP}. As b increases from -0.3 to -0.255, an increase in the asset price and greater borrowing capacity allows the planner to issue more debt (Figure 4), which is induced in the regulated CE through a greater debt subsidy (i.e., $\mathrm{d}b'_x/\mathrm{d}\tau_x > 0$ and $\mathrm{d}q_x/\mathrm{d}\tau_x < 0$ in the context of Proposition 5) that reaches around 40% (τ falls to -40%). As b increases further from -0.255 to -0.23, the optimal debt issuance starts to fall (b'_x increases), and so does the optimal subsidy, reaching zero when the collateral constraint becomes slack.

Figure 2 illustrates how the optimal tax on debt and its macroprudential and ex post components given by (32)–(34) are used around financial crisis episodes. Specifically, we simulate the DE for 101,000 periods, drop the first 1,000, and identify the dates at which the current account $ca_t = b_{t+1} - b_t$ exceeds its two standard deviations, i.e., $ca_t > \overline{ca} \equiv 2\hat{\sigma}(ca_t)$, which indicates a significant capital outflow (Bianchi and Mendoza, 2018). Each such date t indicates a financial crisis event. We then extract the DE states $x_t = (b_t, s_t)$ in a four-year window around each crisis, evaluate the tax functions at these states, and compute an average over all crises paths. Hence, the paths provide the values of the tax that would have to be applied if the policymaker intervened under discretion at a specific date of the crisis window, directly reflecting the policy functions in Figure 1.

Figure 2: Optimal tax around financial crises



Notes: Each line is an average across all financial crisis events. The optimal tax function and its "ex ante" (macroprudential) and ex post components given by (32)–(34) are evaluated at the DE states observed during the crises.

As the economy gets closer to a financial crisis, the macroprudential component increases by 2.3 percentage points, on average, from $\tau_{-4}^{\mathrm{MP}} = 4.4\%$ to $\tau_{-1}^{\mathrm{MP}} = 6.7\%$. When a financial crisis occurs, $\tau_0^{\mathrm{MP}} = 0$, and the policy response is driven by the ex-post intervention that averages at $\tau_0^{\mathrm{EP}} = -30\%$. Although a financial crisis occurs only if the collateral constraint is binding, the converse is not true: the constraint may be binding, but the capital outflow is not large enough to qualify as a crisis. Consequently, the ex post component can be slightly negative before and after a crisis, which explains a slight discrepancy between τ and τ^{MP} at the start and end of the crisis window.

5.3 Restricted optimal time-consistent policy

In this subsection, we compute a restricted optimal time-consistent tax on debt under the additional constraint

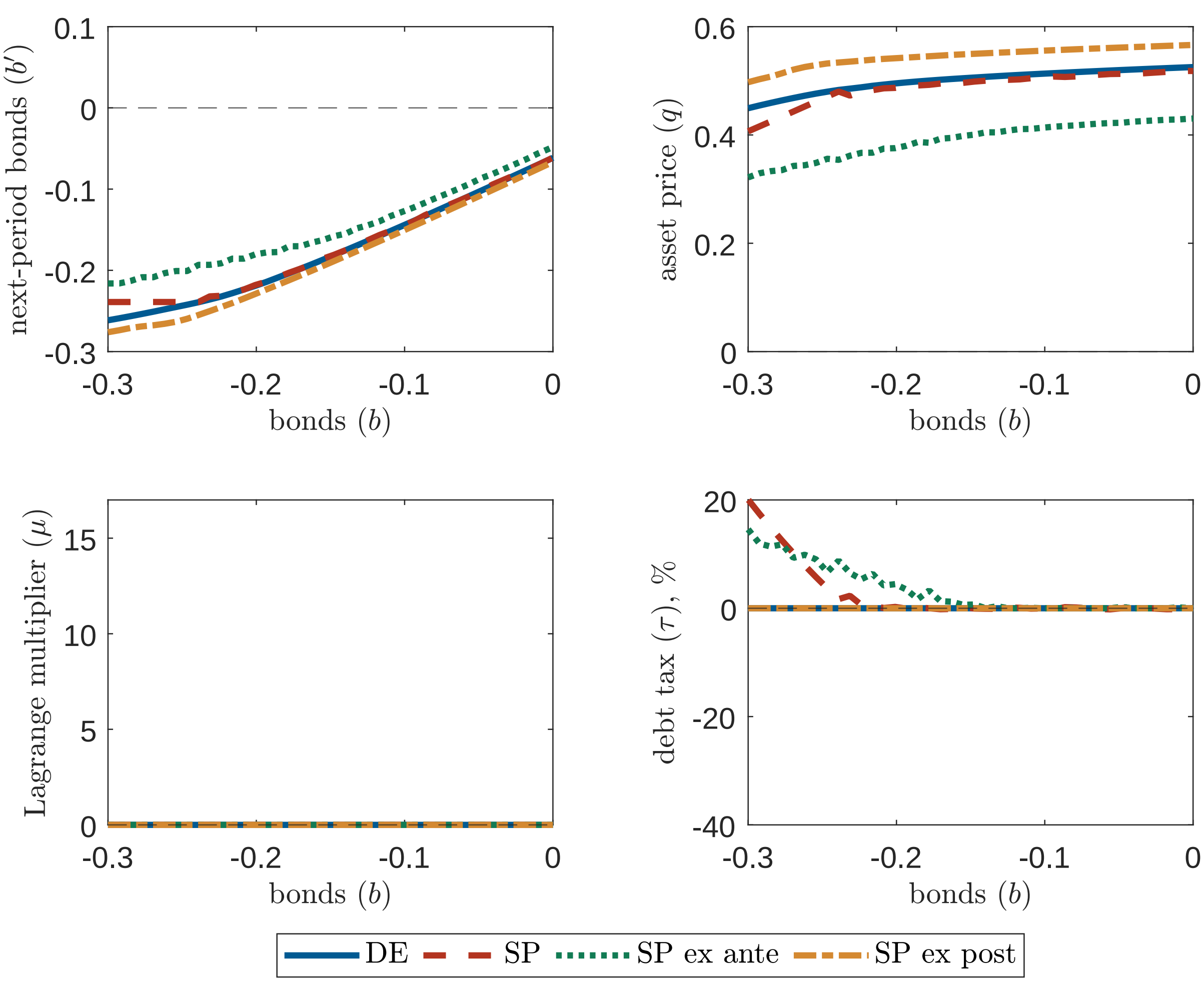
$$\underline{\tau} \leq \tau(x) \leq \overline{\tau}, \qquad \text{for all } x \in X, \tag{40}$$

where $\underline{\tau} \leq 0$ potentially restricts the ex post (subsidy) component, while $\overline{\tau} \geq 0$ may restrict the macroprudential (tax) component. We thus study the MPE of a policy game in Proposition 6 where policymakers are, in addition, constrained by (40). The latter implies that the resulting MPE is *not* constrained efficient, provided (40) binds in some states. Since $\theta > 0$, the resulting MPE, if it exists, is generally unique. Numerically, we impose (40) in the policymaker's best response by backing out τ from (28).

5.3.1 Policy functions and financial crises dynamics

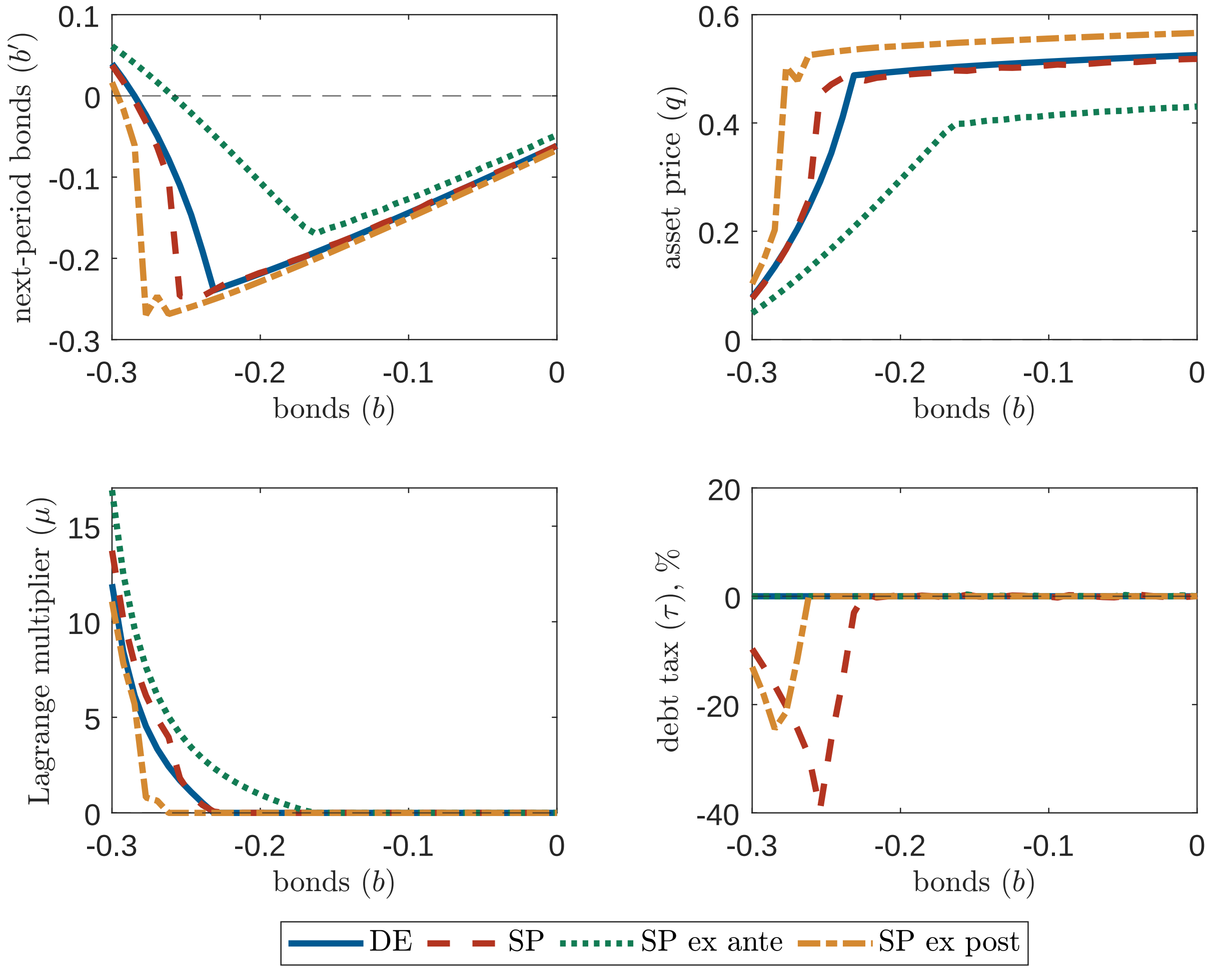
First, we compare the policy functions for next-period bond holdings b', the asset price q, agent's Lagrange multiplier μ, and tax on debt τ in the good (Figure 3) and bad (Figure 4) states (defined as in Figure 1) across the DE, baseline optimal time-consistent policy ("SP"), the optimal policy that allows only a nonnegative tax $\tau_x \geq 0$ ("SP ex ante"), and optimal policy that allows only a subsidy $\tau_x \leq 0$ ("SP ex post").

Figure 3: Policy functions in the good state



Notes: "good state" = (average z, high R, high κ). "SP" corresponds to the unrestricted optimal time-consistent policy ($\underline{\tau} = -\infty$ and $\overline{\tau} = +\infty$), "SP ex ante" to $\underline{\tau} = 0$ and $\overline{\tau} = +\infty$, and "SP ex post" to $\underline{\tau} = -27\%$ and $\overline{\tau} = 0$, where -27% is the lowest $\underline{\tau}$ (up to 1%) such that the MPE exists.

Figure 4: Policy functions in the bad state



Notes: "bad state" = (average z, high R, low κ). "SP" corresponds to the unrestricted optimal time-consistent policy ($\underline{\tau} = -\infty$ and $\overline{\tau} = +\infty$), "SP ex ante" to $\underline{\tau} = 0$ and $\overline{\tau} = +\infty$, and "SP ex post" to $\underline{\tau} = -27\%$ and $\overline{\tau} = 0$, where -27% is the lowest $\underline{\tau}$ (up to 1%) such that the MPE exists.

Consider the good state (Figure 3). In all equilibria, the collateral constraint is slack independently of the level of bond holdings. There is slightly more saving in the SP allocation compared to the DE when current debt is sufficiently large, induced by the positive debt tax. There is significantly more saving in the "SP ex ante" equilibrium, induced by a broader application of the tax on debt in an effort to decrease the probability of a binding constraint in the next period, reflecting the nonavailability of the debt subsidy in the bad state. Conversely, there is more borrowing in the "SP ex post" equilibrium, induced by the nonavailability of a (positive) debt tax in the current state and the use of debt subsidy in the bad states that may occur in next periods. Corresponding to the differences in next-period bond holdings b' are the differences in the asset price functions: more saving in the SP and "SP ex ante" equilibria compared to the DE is associated with lower asset prices in the former compared to the latter. Conversely, asset prices are higher in the "SP ex post" equilibrium compared to the DE. This is consistent with the discussion following Proposition 5, and in particular with Corollaries 2 and 3.

Consider now the bad state (Figure 4). The unconstrained policy (SP) subsidizes debt issuance in the binding region, which is accompanied by higher asset prices due to lower marginal utility of consumption. There is even more debt and higher asset prices in the "SP ex post" equilibrium, but the debt subsidy has smaller magnitude than in the SP equilibrium. The latter follows from the fact that positive taxes are not allowed in the good states, and both debt and asset prices are high in those states. Due to higher asset prices, the binding region is the smallest in the "SP ex post" equilibrium. Conversely, lack of debt subsidies in the "SP ex ante" equilibrium leads to lower debt, lower asset prices, and a larger binding region than in the DE.

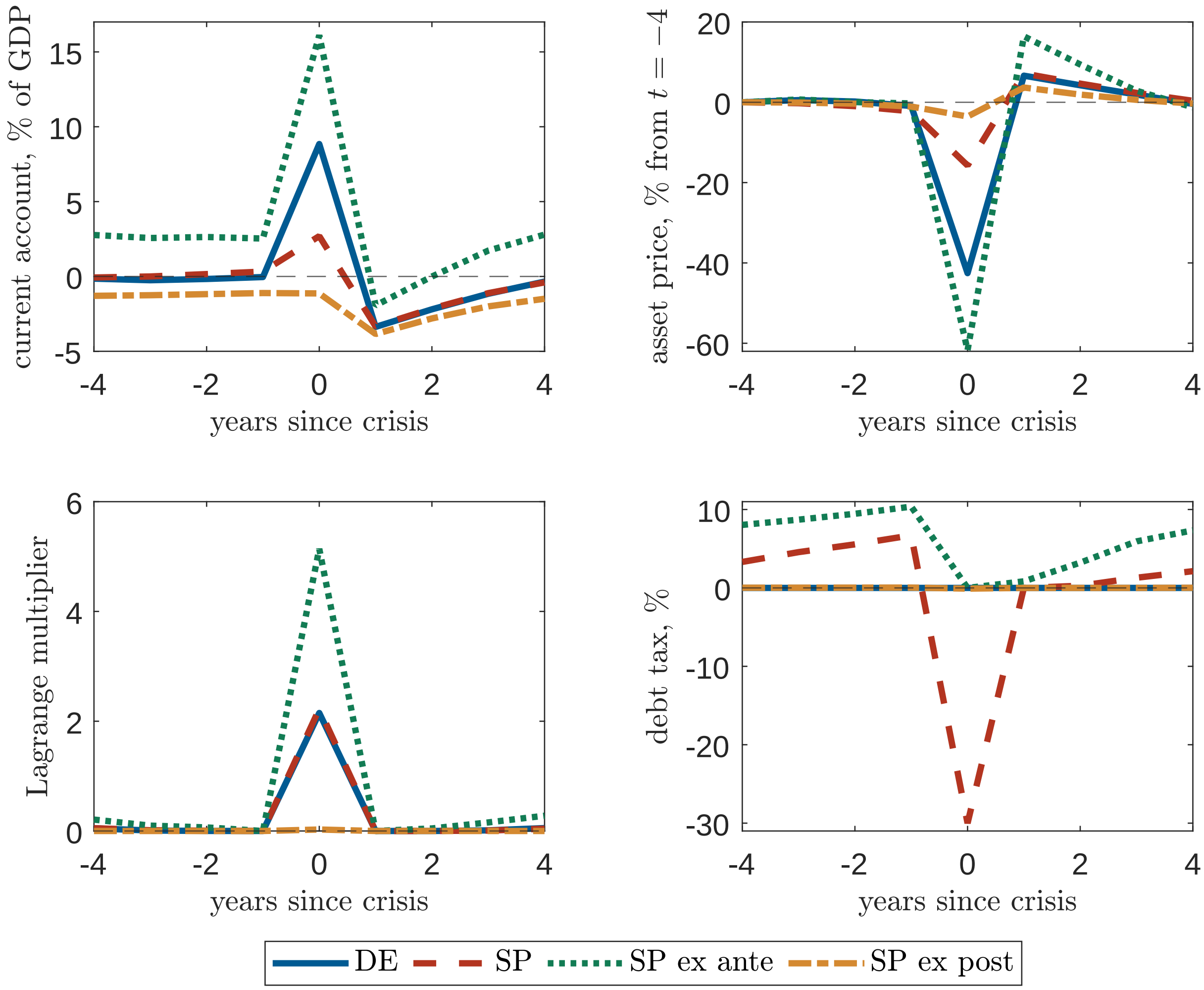
Figure 5 compares the dynamics of the current account (in % of output), asset price, agent's Lagrange multiplier, and tax on debt across alternative equilibria around financial crises. The ex post debt subsidy provided in the SP equilibrium mitigates financial crises by significantly decreasing the capital outflow and the fall in the asset price compared to the DE. The probability of a financial crisis decreases from 4.02% in the DE to 0.01% in the SP equilibrium (see Table 2). Conversely, in the "SP ex ante" equilibrium, the nonavailability of the debt subsidy exacerbates financial crises, leading to a larger capital outflow and greater fall in the asset price compared to the DE. The financial crisis probability decreases, but only to 0.25%. Interestingly, financial crises are almost nonexistent in the "SP ex post" equilibrium at the DE crises states: the collateral constraint is rarely binding in those states, consistent with the smaller binding region in Figure 4, there is virtually no fall in the asset price, no capital outflow, and no need to subsidize debt. Financial crises do occur in the "SP ex post" equilibrium but at greater levels of debt that are not commonly observed in the DE. The financial crisis probability decreases to 1.88%.

5.3.2 Welfare gains

Table 2 reports the welfare gains from alternative equilibria relative to the DE in terms of permanent changes in net consumption, provides selected moments of the corresponding tax functions, and reports the financial crisis probabilities.

The unconstrained optimal time-consistent policy ("SP") induces sizable average welfare gains of more than 0.6%, both with respect to the DE and SP ergodic distributions. The welfare gains are achieved by taxing debt in "good times" and subsidizing debt in "bad times." If either subsidies or (positive) taxes are not available to the policymaker, the optimal time-consistent policy is counterproductive and leads to welfare losses: -0.14% in

Figure 5: Financial crises



Notes: Each line is an average across all financial crisis events. "SP" corresponds to the unrestricted optimal time-consistent policy ($\underline{\tau} = -\infty$ and $\overline{\tau} = +\infty$), "SP ex ante" to $\underline{\tau} = 0$ and $\overline{\tau} = +\infty$, and "SP ex post" to $\underline{\tau} = -27\%$ and $\overline{\tau} = 0$, where -27% is the lowest $\underline{\tau}$ (up to 1%) such that the MPE exists. The policy functions are evaluated at the DE states observed during the crises.

the "SP ex ante" equilibrium and -0.08% in the "SP ex post equilibrium" with respect to the DE ergodic distribution. The welfare losses are smaller if computed with respect to the ergodic distributions of the corresponding equilibria, and there is a marginal welfare gain of 0.01% from the "SP ex post equilibrium" in this case.

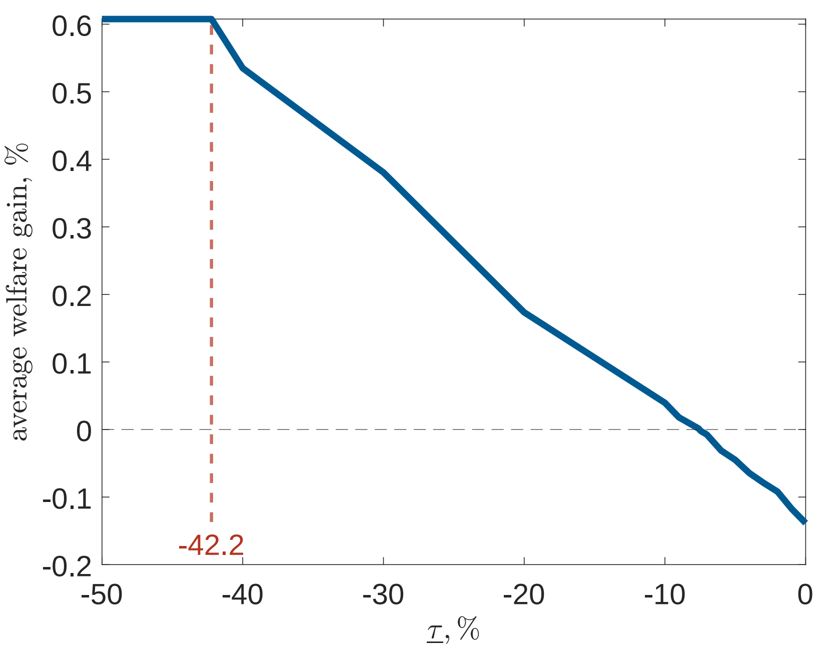
Figure 6 further illustrates the role of the ex post debt subsidy for welfare gains. In this figure, we plot the welfare gains from the optimal time-consistent policy constrained by $\tau(x) \geq \underline{\tau}$ for different values of $\underline{\tau}$ on the horizontal axis. Hence, only the subsidy component is restricted, but taxes can be set as high as needed, i.e., $\overline{\tau} = +\infty$ in (40). If $\underline{\tau} < -42.2\%$ (i.e., $\min(\tau)$ for "SP" in Table 2), the constraint $\tau(x) \geq \underline{\tau}$ is slack, and we obtain the unconstrained

Table 2: Statistics

	DE	SP	SP+	SP−	SP+ best	SP− best
Average welfare gain, initial π	0	0.61	−0.14	−0.08	0.03	0.03
Average welfare gain, final π	0	0.62	−0.03	0.01	0.04	0.03
$\min(\tau)$	0	−42.2	0	−27	0	−6
$\max(\tau)$	0	29.8	20.1	0	0.8	0
$\mathbb{E}(\tau)$	0	2.4	2.7	−0.7	0.8	−0.3
$\Pr(\tau < 0)$	0	8.1	0	30.4	0	5.1
$\Pr\left(ca_t > \overline{ca}^{\mathrm{DE}}\right)$	4.02	0.01	0.25	1.88	3.7	3.57

Notes: All statistics are in %. "SP" corresponds to the unrestricted optimal time-consistent policy ($\underline{\tau} = -\infty$ and $\overline{\tau} = +\infty$), "SP+" to $\underline{\tau} = 0$ and $\overline{\tau} = +\infty$ ("SP ex ante"), "SP−" to $\underline{\tau} = -27\%$ and $\overline{\tau} = 0$ ("SP ex post"), "SP+ best" to $\underline{\tau} = 0$ and $\overline{\tau} = 0.8\%$ (the maximum in the right panel of Figure 7), and "SP− best" to $\underline{\tau} = -6\%$ and $\overline{\tau} = 0$ (the maximum in the left panel of Figure 7). Welfare gains are in permanent net consumption equivalents. "initial π" is the DE ergodic distribution, while "final π" is the ergodic distribution under the corresponding alternative equilibrium. The moments of τ are with respect to the "final π." $\overline{ca}^{\mathrm{DE}}$ is the DE financial crisis threshold.

Figure 6: Welfare gains from optimal time-consistent policy constrained by $\tau(x) \geq \underline{\tau}$

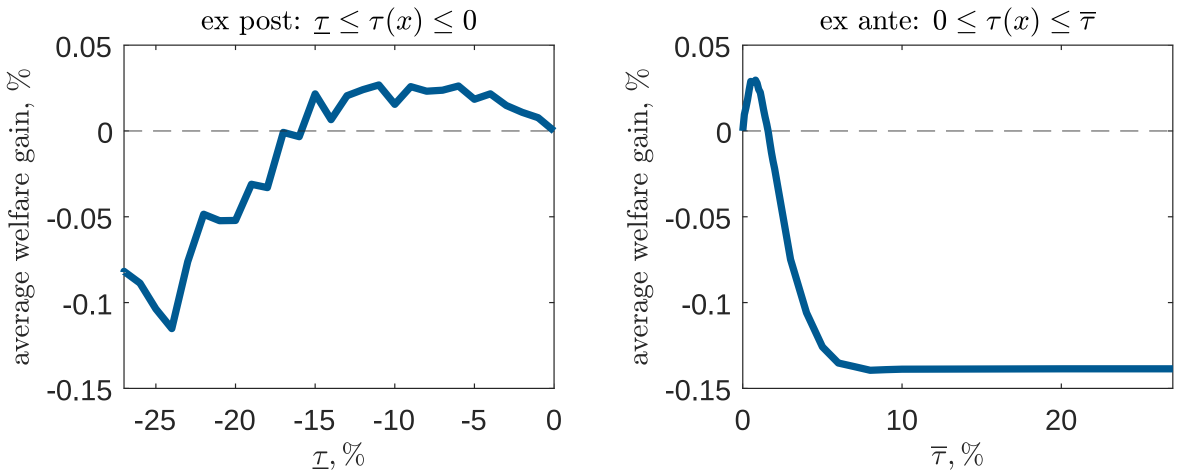


Notes: If $\underline{\tau} < -42.2\%$, the constraint $\tau(x) \geq \underline{\tau}$ is slack, and we obtain the "SP" equilibrium.

"SP" equilibrium. As the tax lower bound $\underline{\tau}$ increases from −42.2%, welfare gains decrease and become negative when $\underline{\tau} \approx -7.5\%$. Hence, substantial ex post interventions are essential for welfare gains from a time-consistent policy.

Interestingly, we find that optimal time-consistent ex ante and ex post policies can be welfare enhancing if we restrict the magnitude of these policies. The left panel of Figure 7

Figure 7: Welfare gains from optimal time-consistent ex post and ex ante policies



Notes: The left panel corresponds to optimal time-consistent policies constrained by $\underline{\tau} \leq \tau(x) \leq 0$ for different values of $\underline{\tau}$. In the right panel, the constraint is $0 \leq \tau(x) \leq \overline{\tau}$ for different values of $\overline{\tau}$.

plots welfare gains from optimal time-consistent policies constrained by $\underline{\tau} \leq \tau(x) \leq 0$ for different values of $\underline{\tau}$ on the horizontal axis. These policies do not allow positive taxes and have an interpretation of "ex post" policies. For $\underline{\tau} \geq -15\%$, there is a welfare gain from an ex post policy, with the maximum welfare gain of around 0.03% when $\underline{\tau} = -6\%$ ("SP$-$ best" in Table 2). Symmetrically, the right panel of Figure 7 considers optimal time-consistent policies constrained by $0 \leq \tau(x) \leq \overline{\tau}$ for different values of $\overline{\tau}$ on the horizontal axis. For $\overline{\tau} \leq 1.5\%$, there is a welfare gain from such "ex ante" policies, with the maximum of also around 0.03% when $\overline{\tau} = 0.8\%$ ("SP+ best" in Table 2). Hence, there is a wider range of welfare-enhancing ex post policies, but welfare gains from such policies are significantly smaller than from the unrestricted optimal time-consistent policy.

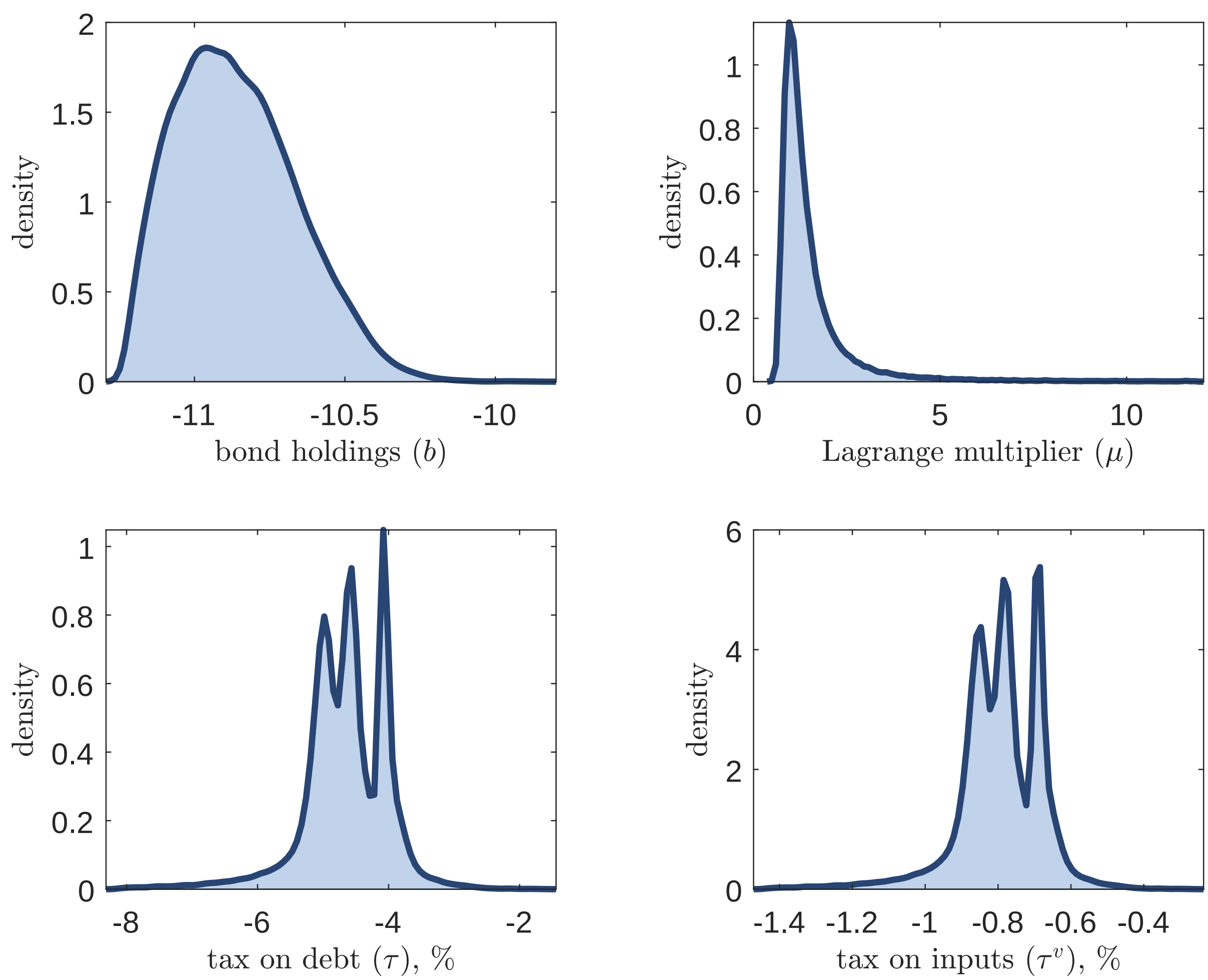
5.4 Implementing unconstrained allocation

In this subsection, we apply Proposition 8 to construct numerically the optimal time-consistent policy $\{(\tau_t, \tau_t^v)\}$ that implements the UE allocation in a regulated CE. Specifically, we compute the UE allocation functions $\left\{\tilde{c}^{\mathrm{UE}}, h^{\mathrm{UE}}, v^{\mathrm{UE}}, b^{\mathrm{UE}}\right\}$ of Definition 2 and the asset price function q^A given by (20), simulate a long sequence of shocks $\{s_t\}$, use (27) to construct a sequence of Lagrange multipliers $\{\mu_t\}$ that are part of a constrained-efficient plan, and use (36) and (38) to back out the corresponding optimal tax rates.

Under the baseline calibration, there is a well-defined stationary distribution corresponding to the UE allocation, consistent with Assumption 1, and the collateral constraint is always violated in the UE, i.e., $\lambda(A) = 1$. However, the condition (22) is not always satisfied. To obtain an example in which the condition (22) holds on X^{UE}, we make two changes relative to the baseline calibration. First, we fix $R_t = \overline{R}$. Hence, the natural borrowing limit remains the same as under the stochastic $\{R_t\}$. Second, we fix $\kappa_t = \kappa^L$. It is possible to allow for a stochastic $\{\kappa_t\}$, but it does not affect the UE allocation. Therefore, in this example, the TFP $\{z_t\}$ is the only source of uncertainty.

Figure 8 displays the empirical distributions of bond holdings, the agent's Lagrange multiplier, and the optimal time-consistent tax rates on debt and inputs that implement the UE allocation in a regulated CE. The distributions are based on a 100,000-period stochastic simulation after a 1,000-period burn-in. The distribution of bond holdings is skewed to the

Figure 8: Unconstrained allocation and optimal time-consistent policy



Notes: Probability density function estimates with a normal kernel based on a 100,000-period simulation after a 1,000-period burn-in. The distribution of μ is truncated above its 99th percentile, while the distributions of τ and τ^v are truncated below their corresponding 1st percentiles.

right, away from the natural borrowing limit. On average, the agent borrows about 46 times more than in the DE (44 times more if $\{R_t\}$ were stochastic). Hence, the DE features *underborrowing* compared to the UE allocation. The condition (22) is always satisfied strictly, so that $\mu_t > 0$, $\tau_t < 0$, and $\tau_t^v < 0$ for all $t > 0$. The distribution of μ has a long right tail, translating into the left tails for τ and τ^v. Most of the mass of τ is concentrated in the $[-7\%, -3\%]$ interval, with three peaks corresponding to the three values of z_t. The distribution of τ^v is very similar to that of τ but scaled down towards zero, with most of the mass of τ^v in the $[-1.3\%, -0.3\%]$ interval. The debt and input *subsidies* successfully eliminate the collateral and working capital constraint distortions.

6 Conclusion

This paper addressed how financial crises should be managed when prevention and resolution policies interact. Using the workhorse collateral-constraint model of Bianchi and Mendoza (2018), we studied the normative problem of a time-consistent planner and showed that multiple constrained-efficient equilibria can arise in this setting. Among them, we identified a welfare-dominant equilibrium that entails the unconstrained allocation, highlighting that eliminating crises should be the benchmark policy objective.

We also demonstrated that the full set of constrained-efficient equilibria can be implemented by combining ex ante macroprudential tools with ex post interventions. Quantitatively, both elements are essential: preventive measures alone can be costly without complementary crisis resolution policies and vice versa.

Returning to Bagehot's classic dictum, our findings suggest that lender-of-last-resort interventions are not merely tools for managing crises after they erupt. They are also integral to making ex ante regulation effective. Effective crisis management, therefore, requires that prevention and resolution be designed jointly, reflecting the institutional and economic structure that shapes financial fragility.

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Appendices

A Proofs

A.1 Proposition 1

Define an operator $T : \mathcal{F}(X) \to \mathcal{F}(X)$ as

$$T(V)(b,s) = \max_{\hat{h},\hat{v},\hat{b}} \left[u\left(zF(1,\hat{h},\hat{v}) - p_v\hat{v} - g(\hat{h}) + b - \frac{\hat{b}}{R} \right) + \beta \mathbb{E}_s V(\hat{b},s') \right],$$

for all $(b,s) = (b,(z,R,\kappa)) \in X$. Since u is strictly increasing,

$$T(V)(b,s) = \max_{\hat{b}} \left[u\left(\max_{\hat{h},\hat{v}} \Big\{ zF(1,\hat{h},\hat{v}) - p_v\hat{v} - g(\hat{h}) \Big\} + b - \frac{\hat{b}}{R} \right) + \beta \mathbb{E}_s V(\hat{b},s') \right].$$

Consider the inner maximization problem

$$f(z) = \max_{\hat{h},\hat{v}} \Big\{ zF(1,\hat{h},\hat{v}) - p_v\hat{v} - g(\hat{h}) \Big\}.$$

Since F is concave and Cobb—Douglas, it is strictly concave in labor and inputs. Also, g is convex. Hence, the unique maximum is described by the first-order conditions (11) and (16). Clearly, the functions h and v depend on z only. To see that they are strictly increasing, suppose for a moment that z can be varied continuously. Then, applying the implicit function theorem,

$$\begin{aligned} \frac{\mathrm{d}h_x}{\mathrm{d}z} &= \frac{F_h(1,h_x,v_x) - \frac{F_{hv}(1,h_x,v_x)}{F_{vv}(1,h_x,v_x)} F_v(1,h_x,v_x)}{g''(h_x) - z\left(F_{hh}(1,h_x,v_x) - \frac{F_{hv}(1,h_x,v_x)^2}{F_{vv}(1,h_x,v_x)} \right)} > 0, \\ \frac{\mathrm{d}v_x}{\mathrm{d}z} &= -\frac{F_v(1,h_x,v_x)}{zF_{vv}(1,h_x,v_x)} - \frac{F_{hv}(1,h_x,v_x)}{F_{vv}(1,h_x,v_x)} \frac{\mathrm{d}h_x}{\mathrm{d}z} > 0, \end{aligned}$$

where the signs follow from our assumptions on F and g that imply $F_h, F_v, F_{hv} > 0$, $F_{hh}, F_{vv} < 0$, $g'' \geq 0$, and $F_{hh} - \frac{F_{hv}^2}{F_{vv}} < 0$, where the latter is due to strict concavity of F in the last two arguments and its Cobb—Douglas form. Note that f is also strictly increasing by the envelope theorem.

The operator T simplifies to

$$T(V)(b,s) = \max_{\hat{b}\in\Gamma(b,s)} \left[u\left(f(z) + b - \frac{\hat{b}}{R} \right) + \beta \mathbb{E}_s V(\hat{b}, s') \right],$$

where $\Gamma(b,s) = (-\infty, R(f(z)+b)] \cap B$, so that $\hat{c} \geq 0$. We have $\Gamma(b,s) \neq \emptyset$ for all $(b,s) \in X$ if and only if $\underline{b} \leq R(f(\underline{z}) + \underline{b})$ for all $R \in [\underline{R}, \overline{R}]$, which is if and only if $\underline{b} \geq -\frac{\overline{R}}{\overline{R}-1} f(\underline{z})$ if $\overline{R} > 1$ and $\underline{b} \leq \frac{\underline{R}}{1-\underline{R}} f(\underline{z})$ if $\underline{R} < 1$. Let $\underline{b}$ satisfy these inequalities strictly, so that $\Gamma(b,s)$ is infinite for all $(b,s) \in X$. Clearly, Γ is compact-valued and continuous. Since S is finite, $X = B \times S$ is compact, so any continuous function on X is bounded by the extreme value theorem. Since u is continuous, it then follows from the maximum theorem that $T : C(X) \to C(X)$, where $C(X) \subset \mathcal{F}(X)$ is the set of all continuous (and thus bounded) functions on X. Then by Theorem 9.6 in Stokey et al. (1989), T has a unique fixed point V—the solution to the Bellman equation.

Since u is strictly concave and continuously differentiable and Γ is convex in b, it follows that V is strictly concave in b, the policy function b' is single-valued and continuous, and V is continuously differentiable in b at interior points whenever $b'(b,s)$ is interior in $\Gamma(b,s)$ (Stokey et al., 1989, Theorems 9.8 and 9.10). Specifically,

$$V_b(b,s) = u'(\tilde{c}(b,s)), \tag{A.1}$$

where $\tilde{c}(b,s) = f(z) + b - \frac{b'(b,s)}{R}$ is the implied policy function for net consumption, and the first-order condition for an interior maximum is

$$u'(\tilde{c}(b,s)) = \beta R \mathbb{E}_s V_b(b'(b,s), s'). \tag{A.2}$$

The Euler equation (17) is obtained by substituting (A.1) into (A.2). Note that $b'(b,s) = R(f(z)+b)$ cannot be optimal since it implies $u'(\tilde{c}(b,s)) = \infty$. Hence, if $b'(b,s)$ is at the boundary of $\Gamma(b,s)$, then it must be at the boundary of B.

Clearly, $\tilde{c}$ is strictly increasing in b if $b'(b,s)$ is at the boundary of B. Since V is strictly concave in b and u is strictly concave, (A.1) implies that $\tilde{c}$ is strictly increasing in b if $b'(b,s) \in \operatorname{int} B$, and then (A.2) implies that b' is injective in b in the same region. Since b' is continuous, it then must be strictly monotone in b whenever $b'(b,s) \in \operatorname{int} B$. It is clear from (A.2) that b' cannot be strictly decreasing in b, hence it is strictly increasing in b.

A.2 Proposition 2

In an MPE, $\mu_x \geq 0$ for all $x \in X$. Hence, given the UE allocation, (14) and (18) imply $q_x \geq q_x^{\mathrm{UE}}$ for all $x \in X$. Substituting (20) into the collateral constraint (10) evaluated at the UE allocation, we obtain $-b_x^{\mathrm{UE}}/R + \theta p_v v_x^{\mathrm{UE}} = \kappa q_x^A \leq \kappa q_x$, which is equivalent to $q_x \geq q_x^A$. Hence, the complementary slackness condition in (15) evaluated at the UE allocation is equivalent to $\mu_x \left(q_x - q_x^A\right) = 0$. Now let $x \in A$ and suppose $x(s^t) \in \operatorname{int} A^c$ for all $t \geq 1$ and $s^t \in S^t$. Then due to (19), we have $q^A(x(s^t)) < q^{\mathrm{UE}}(x(s^t)) \leq q(x(s^t))$, which implies $\mu(x(s^t)) = 0$ for all $t \geq 1$ and $s^t \in S^t$, and thus $q_x = q_x^{\mathrm{UE}}$. But since $x \in A$, we have $q_x = q_x^{\mathrm{UE}} < q_x^A$, which is a contradiction.

A.3 Proposition 3

Since $\left\{\tilde{c}^{\mathrm{UE}}, h^{\mathrm{UE}}, v^{\mathrm{UE}}, b^{\mathrm{UE}}\right\}$ satisfy (9), $\left\{\tilde{c}^{\mathrm{UE}}, h^{\mathrm{UE}}, v^{\mathrm{UE}}, b^{\mathrm{UE}}, q, \mu\right\} \subset \mathcal{F}(X)$ is a constrained-efficient MPE of Definition 3 if and only if

$$-\frac{b_x^{\mathrm{UE}}}{R} + \theta p_v v_x^{\mathrm{UE}} \leq \kappa q_x, \tag{A.3}$$

$$q_x u'\left(\tilde{c}_x^{\mathrm{UE}}\right) = \beta \mathbb{E}_s\left[u'\left(\tilde{c}_{x'}^{\mathrm{UE}}\right)\left(z' F_k\left(1, h_{x'}^{\mathrm{UE}}, v_{x'}^{\mathrm{UE}}\right) + q_{x'}\right) + \mu_{x'} \kappa' q_{x'}\right], \tag{A.4}$$

$$0 = \mu_x\left(\kappa q_x + \frac{b_x^{\mathrm{UE}}}{R} - \theta p_v v_x^{\mathrm{UE}}\right), \qquad \mu_x \geq 0, \tag{A.5}$$

with $x' = \left(b_x^{\mathrm{UE}}, (z', R', \kappa')\right)$, for all $x = (b, s) = (b, (z, R, \kappa)) \in X$. The "only if" is by definition, while "if" is since $\left\{\tilde{c}^{\mathrm{UE}}, h^{\mathrm{UE}}, v^{\mathrm{UE}}, b^{\mathrm{UE}}\right\}$ is optimal in the less constrained problem of Definition 2.

Proposition 2 requires $q_x \geq q_x^A > q_x^{\mathrm{UE}}$ for all $x \in A$, which requires $\mu(x(s^t)) > 0$ for some $t \geq 1$ and $s^t \in S^t$. In turn, by Proposition 2, $\mu(x(s^t)) > 0$ implies $q(x(s^t)) = q^A(x(s^t))$. We can, therefore, set $q_x = q_x^A$ for all $x \in A$, which makes (A.3) hold with equality for all $x \in A$. Let $\mathcal{F}_A(X) = \{q \in \mathcal{F}(X) \mid q_x = q_x^A \text{ for all } x \in A\}$. If $\mu_x \geq 0$ for all $x \in X$, any fixed point $q \in \mathcal{F}_A(X)$ of (A.4) makes (A.3) hold for all $x \in A^c$, which follows from $q_x \geq q_x^{\mathrm{UE}} \geq q_x^A$ for all $x \in A^c$ by Proposition 2 and (19). We are left to construct $\mu \in \mathcal{F}(X)$ consistent with (A.5) for all $x \in X$ and (A.4) for all $x \in A$.

Given $q \in \mathcal{F}_A(X)$, define $\mu^q \in \mathcal{F}(X)$ such that $\left\{\left\{\mu^q_{x(s^t)}\right\}_{s^t \in S^t}\right\}_{t=1}^n$ satisfies

$$q^A_x = \sum_{t=1}^n \beta^t \sum_{s^t \in S^t} \Pr\big(s^t \mid s\big) \prod_{i=1}^{t-1} \mathbf{1}_{A^c}(x(s^i)) \frac{u'\left(\tilde{c}^{\mathrm{UE}}_{x(s^t)}\right)}{u'\left(\tilde{c}^{\mathrm{UE}}_x\right)} \times \left[z_t F_k\left(1, h^{\mathrm{UE}}_{x(s^t)}, v^{\mathrm{UE}}_{x(s^t)}\right) + \mathbf{1}_A(x(s^t)) \left(1 + \frac{\mu^q_{x(s^t)} \kappa_t}{u'\left(\tilde{c}^{\mathrm{UE}}_{x(s^t)}\right)}\right) q^A_{x(s^t)}\right] + \beta^n \sum_{s^n \in S^n} \Pr(s^n \mid s) \prod_{i=1}^n \mathbf{1}_{A^c}(x(s^i)) \frac{u'\left(\tilde{c}^{\mathrm{UE}}_{x(s^n)}\right)}{u'\left(\tilde{c}^{\mathrm{UE}}_x\right)} q_{x(s^n)}, \quad \text{(A.6)}$$

for all $x = (b, s) \in A$, where $n \in [1, \infty)$ is such that

$$\sum_{t=1}^n \left|\left\{s^t \in S^t \mid x(s^t) \in A \text{ and } x(s^i) \in A^c \text{ for all } i \in [1, t)\right\}\right| \geq \left|\left\{\hat{x} \in A \mid b^{\mathrm{UE}}(\hat{x}) = b^{\mathrm{UE}}(x)\right\}\right|,$$

where $|\cdot|$ denotes the cardinality of a set, and $\mu^q(\cdot) = 0$ otherwise. Note that (A.6) is (A.4) iterated forward.

Several comments are in order. First, since $\lambda(A) > 0$, we are guaranteed to have an infinite number of $t \geq 1$ and $s^t \in S^t$ such that $x(s^t) \in A$. Therefore, we can indeed affect the right-hand side of (A.6) by varying μ^q. Second, for each $x \in A$, there may be several $\hat{x} \in A$ such that $b^{\mathrm{UE}}(\hat{x}) = b^{\mathrm{UE}}(x)$. Therefore, making μ^q consistent with (A.6) at $x \in A$ requires solving a system of equations corresponding to $\left\{\hat{x} \in A \mid b^{\mathrm{UE}}(\hat{x}) = b^{\mathrm{UE}}(x)\right\}$. Since b^{UE} is strictly increasing in b by Proposition 1, $\left|\left\{\hat{x} \in A \mid b^{\mathrm{UE}}(\hat{x}) = b^{\mathrm{UE}}(x)\right\}\right| \leq |S| < \infty$, so there is a finite number of equations. The system is linear in $\left\{\mu^q_{x(s^t)}\right\}$. For this linear system to have a solution, n needs to be large enough. Due to ergodicity (Assumption 1) and continuity of $\left\{\tilde{c}^{\mathrm{UE}}, h^{\mathrm{UE}}, v^{\mathrm{UE}}, b^{\mathrm{UE}}\right\}$ (Proposition 1), we can choose n such that the number of variables $\left\{\mu^q_{x(s^t)}\right\}$ is at least as great as the number of equations and the rank condition of Rouché—Capelli theorem holds. Third, since b^{UE} is strictly increasing in b, as we vary $x \in A$, the set $\left\{\mu^q_{x(s^t)}\right\}$ will vary as well, allowing to construct a function $\mu^q \in \mathcal{F}(X)$. By imposing $\mu^q(\cdot) = 0$ at all other points unrestricted by the construction above, we ensure that (A.5) holds at all $x \in A^c$.

Define an operator $T : \mathcal{F}_A(X) \to \mathcal{F}_A(X)$ as

$$T(q)(x) = \mathbf{1}_A(x) q_x^A \\ + \mathbf{1}_{A^c}(x) \mathbb{E}_s \left\{ \beta \frac{u'\left(\tilde{c}_{x'}^{\mathrm{UE}}\right)}{u'\left(\tilde{c}_x^{\mathrm{UE}}\right)} \left[z' F_k\left(1, h_{x'}^{\mathrm{UE}}, v_{x'}^{\mathrm{UE}}\right) + \left(1 + \frac{\max\{\mu_{x'}^q, 0\}\kappa'}{u'\left(\tilde{c}_{x'}^{\mathrm{UE}}\right)}\right) q_{x'} \right] \right\},$$

with $x' = \left(b_x^{\mathrm{UE}}, (z', R', \kappa')\right)$, for all $x = (b, s) \in X$. Let $q = T(q) \in \mathcal{F}_A(X)$ with the corresponding $\mu^q \in \mathcal{F}(X)$ and define $\mu \in \mathcal{F}(X)$ as $\mu_x = \max\{\mu_x^q, 0\}$. Such $\{q, \mu\}$ make (A.3) and (A.5) hold on X and (A.4) on A^c. If $\mu_x^q \geq 0$ for all $x \in A$, then (A.4) holds on A, otherwise holding approximately on A. Hence, $\left\{\tilde{c}^{\mathrm{UE}}, h^{\mathrm{UE}}, v^{\mathrm{UE}}, b^{\mathrm{UE}}, q, \mu\right\}$ is a candidate unconstrained MPE of Definition 4, and it is an unconstrained MPE if $\mu_x^q \geq 0$ for all $x \in A$.

If $\lambda(A) = 1$, a significant simplification is achieved if we restrict attention to $X = X^{\mathrm{UE}}$. In this case, there is no need to find a fixed point q of T, since $\lambda(A^c) = 0$, so that (A.6) simplifies to (21), and $q = q^A$ on X^{UE}.

A.4 Proposition 4

The UE allocation $\left\{\left(\tilde{c}_t^{\mathrm{UE}}, h_t^{\mathrm{UE}}, v_t^{\mathrm{UE}}, b_{t+1}^{\mathrm{UE}}\right)\right\}_{t=0}^{\infty}$ satisfies (23) with equality. Since $x_0 \in X^{\mathrm{UE}}$, $x(s^t) \in X^{\mathrm{UE}}$ for all $t \geq 0$ and $s^t \in S^t$ due to Assumption 1. Since $\lambda(A) = 1$, (24) is violated at the UE allocation when prices are $\left\{q_t^{\mathrm{UE}}\right\}_{t=0}^{\infty}$ a.e. on X^{UE}. Setting $\{q_t\}_{t=0}^{\infty} = \left\{q_t^A\right\}_{t=0}^{\infty}$ makes (24) hold with equality on X^{UE}. Setting $\{\mu_t\}_{t=0}^{\infty}$ according to (27) makes (25) hold given the allocation and prices. Finally, $\mu_0 = 0$ and (22) imply $\mu_t(s^t) \geq 0$ for all $t \geq 0$ and $s^t \in S^t$, satisfying (26). Therefore, the proposed plan is feasible in the planning problem of Definition 5. Since the proposed plan entails the UE allocation, it is optimal. Proposition 1 implies that the UE allocation is unique, so any constrained-efficient plan entails the UE allocation.

Consider restarting the planning problem of Definition 5 at some $\tau > 0$ and $s^\tau \in S^\tau$. The continuation of the original plan is feasible, and it is optimal to choose a plan that entails the UE allocation. There is no incentive to deviate from $\left\{\left\{\left(q_t^A(s^t), \mu_t(s^t)\right)\right\}_{s^t \in S^t | s^\tau}\right\}_{t=\tau}^{\infty}$. Consequently, any constrained-efficient plan is time consistent.

A.5 Proposition 5

For any $x \in \operatorname{int} X_u^\tau \cup X_c^\tau$, (9), (11), and (12) imply

$$\mathrm{d}\tilde{c}_x + \frac{1}{R}\,\mathrm{d}b'_x = \frac{\theta p_v \mu_x}{u'(\tilde{c}_x)}\,\mathrm{d}v_x\,, \tag{A.7}$$

$$g''(h_x)\,\mathrm{d}h_x = zF_{hh}(1,h_x,v_x)\,\mathrm{d}h_x + zF_{hv}(1,h_x,v_x)\,\mathrm{d}v_x\,, \tag{A.8}$$

$$\theta p_v\left(\frac{1}{u'(\tilde{c}_x)}\,\mathrm{d}\mu_x - \mu_x\frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)^2}\,\mathrm{d}\tilde{c}_x\right) = zF_{hv}(1,h_x,v_x)\,\mathrm{d}h_x + zF_{vv}(1,h_x,v_x)\,\mathrm{d}v_x\,. \tag{A.9}$$

Moreover, (14) and (28) imply

$$q_x u''(\tilde{c}_x)\,\mathrm{d}\tilde{c}_x + u'(\tilde{c}_x)\,\mathrm{d}q_x = \beta\mathbb{E}_s\frac{\partial}{\partial b'_x}\left[u'(\tilde{c}_{x'})\Big(z'F_k(1,h_{x'},v_{x'}) + q_{x'}\Big) + \mu_{x'}\kappa' q_{x'}\right]\mathrm{d}b'_x\,, \tag{A.10}$$

$$u''(\tilde{c}_x)\,\mathrm{d}\tilde{c}_x = (1+\tau_x)\left[\beta R\mathbb{E}_s\left(u''(\tilde{c}_{x'})\frac{\partial\tilde{c}_{x'}}{\partial b'_x}\right)\mathrm{d}b'_x + \mathrm{d}\mu_x\right] + \frac{u'(\tilde{c}_x)}{1+\tau_x}\,\mathrm{d}\tau_x\,, \tag{A.11}$$

provided all partial derivatives exist.

If $x \in \operatorname{int} X_u^\tau$, we have $\mathrm{d}\mu_x = \mu_x = 0$, (A.8) and (A.9) imply $\mathrm{d}h_x = \mathrm{d}v_x = 0$, and (A.7) implies $\mathrm{d}\tilde{c}_x = -\frac{1}{R}\,\mathrm{d}b'_x$. Hence, (A.11) implies (29) and (A.10) implies (30).

If $x \in X_c^\tau$, (15) implies

$$0 = \kappa\,\mathrm{d}q_x + \frac{1}{R}\,\mathrm{d}b'_x - \theta p_v\,\mathrm{d}v_x\,. \tag{A.12}$$

If $\theta = 0$, (A.8) and (A.9) imply $\mathrm{d}h_x = \mathrm{d}v_x = 0$ and (A.7) implies $\mathrm{d}\tilde{c}_x = -\frac{1}{R}\,\mathrm{d}b'_x$. Hence, (A.10) and (A.12) imply $\mathrm{d}b'_x = \mathrm{d}q_x = 0$, and thus $\mathrm{d}\tilde{c}_x = 0$, and (A.11) implies

$$\mathrm{d}\mu_x = -\frac{u'(\tilde{c}_x)}{(1+\tau_x)^2}\,\mathrm{d}\tau_x\,.$$

If $\theta > 0$, (A.12), (A.8), (A.7), and (A.9) imply

$$\mathrm{d}v_x = \frac{1}{\theta p_v}\left(\kappa\,\mathrm{d}q_x + \frac{1}{R}\,\mathrm{d}b'_x\right), \tag{A.13}$$

$$\mathrm{d}h_x = \frac{zF_{hv}(1,h_x,v_x)}{g''(h_x) - zF_{hh}(1,h_x,v_x)}\,\mathrm{d}v_x\,, \tag{A.14}$$

$$\mathrm{d}\tilde{c}_x = \frac{\mu_x}{u'(\tilde{c}_x)}\left(\kappa\,\mathrm{d}q_x + \frac{1}{R}\,\mathrm{d}b'_x\right) - \frac{1}{R}\,\mathrm{d}b'_x\,, \tag{A.15}$$

$$\begin{aligned}\mathrm{d}\mu_x = {}& \frac{u'(\tilde{c}_x)}{(\theta p_v)^2}\left[\frac{(zF_{hv}(1,h_x,v_x))^2}{g''(h_x) - zF_{hh}(1,h_x,v_x)} + zF_{vv}(1,h_x,v_x)\right]\left(\kappa\,\mathrm{d}q_x + \frac{1}{R}\,\mathrm{d}b'_x\right)\\ & + \mu_x\frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)}\,\mathrm{d}\tilde{c}_x\,. \end{aligned}\tag{A.16}$$

Consequently, (A.10) implies

$$\mathrm{d}q_x = \Psi_x^{q(\tau)}\,\mathrm{d}b'_x\,, \tag{A.17}$$

where

$$\Psi_x^{q(\tau)} \equiv \frac{\frac{1}{R} q_x u''(\tilde{c}_x)\left(1 - \frac{\mu_x}{u'(\tilde{c}_x)}\right) + \beta \mathbb{E}_s \frac{\partial}{\partial b'_x}\left[u'(\tilde{c}_{x'})\Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'}\Big) + \mu_{x'}\kappa' q_{x'}\right]}{u'(\tilde{c}_x) + \mu_x \kappa q_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)}},$$

and (A.11) implies

$$\mathrm{d}b'_x = \frac{u'(\tilde{c}_x)}{\Psi_x^{b'(\tau)}(1+\tau_x)^2}\,\mathrm{d}\tau_x\,, \tag{A.18}$$

where

$$\Psi_x^{b'(\tau)} \equiv -\frac{u''(\tilde{c}_x)}{R}\left(\frac{1}{1+\tau_x} - \frac{\mu_x}{u'(\tilde{c}_x)}\right) - \beta R \mathbb{E}_s\left(u''(\tilde{c}_{x'})\frac{\partial \tilde{c}_{x'}}{\partial b'_x}\right) + \left(\kappa \Psi_x^{q(\tau)} + \frac{1}{R}\right)$$
$$\times \left\{\mu_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)}\left(\frac{1}{1+\tau_x} - \frac{\mu_x}{u'(\tilde{c}_x)}\right) - \frac{u'(\tilde{c}_x)}{(\theta p_v)^2}\left[\frac{(zF_{hv}(1,h_x,v_x))^2}{g''(h_x) - zF_{hh}(1,h_x,v_x)} + zF_{vv}(1,h_x,v_x)\right]\right\}.$$

A.6 Proposition 6

The regulated CE of Definition 6 is described by (9)–(12), (14), (15), and (28). Clearly, (28) can be used to back out τ given the other functions. We will show that (11), (12), and (15) are slack as constraints in the current policymaker's best response to the future policymaker's decision rules. Similar to Bianchi and Mendoza (2018, Appendix A.1, Proposition II), who assume $\theta > 0$, consider the best response in a relaxed problem given by

$$V(b,s) = \max_{\hat{c},\hat{h},\hat{v},\hat{b},\hat{q}}\left[u(\hat{c}) + \beta \mathbb{E}_s V(\hat{b}, s')\right]$$

subject to

$$\hat{c} + \frac{\hat{b}}{R} \le zF(1,\hat{h},\hat{v}) - p_v \hat{v} - g(\hat{h}) + b,$$
$$-\frac{\hat{b}}{R} + \theta p_v \hat{v} \le \kappa \hat{q},$$
$$\hat{q} u'(\hat{c}) = \beta \mathbb{E}_s\left[u'(\tilde{c}_{x'})\Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'}\Big) + \mu_{x'}\kappa' q_{x'}\right],$$

with $x' = (\hat{b}, (z', R', \kappa'))$, for all $x = (b, s) = (b, (z, R, \kappa)) \in X$. The corresponding Lagrangian is

$$\mathcal{L} = u(\hat{c}) + \beta \mathbb{E}_s V(\hat{b}, s') + \hat{\lambda} \left(zF(1, \hat{h}, \hat{v}) - p_v \hat{v} - g(\hat{h}) + b - \hat{c} - \frac{\hat{b}}{R} \right)$$
$$+ \hat{\mu}^{\mathrm{SP}} \left(\kappa \hat{q} + \frac{\hat{b}}{R} - \theta p_v \hat{v} \right) + \hat{\xi} \left\{ \beta \mathbb{E}_s \left[u'(\tilde{c}_{x'}) \Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'} \Big) + \mu_{x'} \kappa' q_{x'} \right] - \hat{q} u'(\hat{c}) \right\}.$$

The first-order conditions for $\hat{c}$, $\hat{h}$, $\hat{v}$, $\hat{q}$ and the complementary slackness conditions in an MPE are, respectively,

$$\lambda_x = u'(\tilde{c}_x) - \xi_x q_x u''(\tilde{c}_x), \tag{A.19}$$

$$0 = \lambda_x \Big(zF_h(1, h_x, v_x) - g'(h_x) \Big), \tag{A.20}$$

$$0 = \lambda_x \Big(zF_v(1, h_x, v_x) - p_v \Big) - \mu_x^{\mathrm{SP}} \theta p_v, \tag{A.21}$$

$$\xi_x = \mu_x^{\mathrm{SP}} \frac{\kappa}{u'(\tilde{c}_x)}, \tag{A.22}$$

$$0 = \mu_x^{\mathrm{SP}} \left(\kappa q_x + \frac{b'_x}{R} - \theta p_v v_x \right), \qquad \mu_x^{\mathrm{SP}} \geq 0. \tag{A.23}$$

Since $\mu_x^{\mathrm{SP}} \geq 0$ by (A.23), $\xi_x \geq 0$ by (A.22), and $\lambda_x > 0$ by (A.19). Hence, (A.20) is equivalent to (11).

If $\theta = 0$, (A.21) is equivalent to (12), which means that the set of MPE in the optimal policy problem is equivalent to the set of constrained-efficient MPE of Definition 3. Specifically, the current policymaker, taking as given the future policymaker's decision rule μ, can set $\hat{\mu}_x = \mu_x$ today to satisfy (15). There generally exist multiple such μ, and thus multiple MPE (Remark 3).

If $\theta > 0$, rearranging (A.21), we obtain

$$\left(1 + \theta \frac{\mu_x^{\mathrm{SP}}}{\lambda_x} \right) p_v = zF_v(1, h_x, v_x). \tag{A.24}$$

If

$$\mu_x = \frac{u'(\tilde{c}_x)}{\lambda_x} \mu_x^{\mathrm{SP}},$$

(A.24) is equivalent to (12) and (A.23) is equivalent to (15). This selects a specific MPE of Definition 3 that corresponds to μ satisfying (31), having used (A.19) and (A.22). If it were an unconstrained MPE of Definition 4, (16) would have to hold, requiring $\mu_x^{\mathrm{SP}} = \mu_x = 0$ for all $x \in X$, and thus $q = q^{\mathrm{UE}}$, which could be consistent with (10) only if $\lambda(A) = 0$.

The first-order condition for $\hat{b}$ in an MPE is

$$0 = \beta \mathbb{E}_s V_b(b'_x, s') - \frac{\lambda_x}{R} + \frac{\mu_x^{\mathrm{SP}}}{R} + \xi_x \beta \mathbb{E}_s \frac{\partial}{\partial b'_x} \left[u'(\tilde{c}_{x'}) \Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'} \Big) + \mu_{x'} \kappa' q_{x'} \right],$$

with $x' = (b'_x, (z', R', \kappa'))$, provided the partial derivative $\partial / \partial b'_x$ above exists. The envelope condition in an MPE is $V_b(b, s) = \lambda(b, s)$. Combining this with (A.19) and (A.22), we obtain

$$\begin{aligned} u'(\tilde{c}_x) &= \beta R \mathbb{E}_s \left(u'(\tilde{c}_{x'}) - \mu_{x'}^{\mathrm{SP}} \kappa' q_{x'} \frac{u''(\tilde{c}_{x'})}{u'(\tilde{c}_{x'})} \right) \\ &+ \mu_x^{\mathrm{SP}} \left\{ 1 + \kappa q_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)} + \frac{\kappa \beta R}{u'(\tilde{c}_x)} \mathbb{E}_s \frac{\partial}{\partial b'_x} \left[u'(\tilde{c}_{x'}) \Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'} \Big) + \mu_{x'} \kappa' q_{x'} \right] \right\}. \end{aligned} \tag{A.25}$$

The tax τ_x that makes (A.25) equivalent to (28) is given by (32)–(34).

A.7 Proposition 7

The corresponding regulated CE is described by the same conditions as in Definition 6, except (12) is replaced by

$$\left(1 + \tau_x^v + \theta \frac{\mu_x}{u'(\tilde{c}_x)} \right) p_v = z F_v(1, h_x, v_x). \tag{A.26}$$

The policymaker's best response is characterized by exactly the same relaxed problem as in Appendix A.6. Indeed, the solution to that problem implies that (11) holds in an MPE, (A.26) can be used to back out τ^v given the other functions, and μ can be chosen to satisfy (15) as explained in Appendix A.6 for the case $\theta = 0$ (in the current problem, the argument applies to any $\theta \geq 0$). The latter implies that the set of MPE in the current optimal policy problem is equivalent to the set of constrained-efficient MPE of Definition 3.

Since the relaxed policy problem is equivalent to that in Appendix A.6, so is the generalized Euler equation (A.25) that describes the MPE allocation of bond holdings, and thus (32)–(34) continue to hold. The tax τ_x^v that makes (A.24) equivalent to (A.26) is given by (37), having used (A.19) and (A.22). The policy that implements an unconstrained MPE of Definition 4 is obtained by imposing $\mu_x^{\mathrm{SP}} = 0$ for all $x \in X$ in (32)–(34) and (37), which gives (36) and (38).

A.8 Proposition 8

A Ramsey-optimal policy $\{(\tau_t, \tau_t^v)\}_{t=0}^{\infty}$ and the associated allocation $\{(\tilde{c}_t, h_t, v_t, b_{t+1})\}_{t=0}^{\infty}$, prices $\{q_t\}_{t=0}^{\infty}$, and Lagrange multipliers $\{\mu_t\}_{t=0}^{\infty}$ solve the problem in Definition 5 modified by adding $\{(\tau_t, \tau_t^v)\}_{t=0}^{\infty}$ to the set of controls and adding the additional constraints (11), (28), and (A.26) stated in sequential notation. The augmented set of constraints describes a regulated sequential competitive equilibrium given $\{(\tau_t, \tau_t^v)\}_{t=0}^{\infty}$. Applying the primal approach, we can use (28) and (A.26) to back out τ_t and τ_t^v, respectively, as

$$\tau_t = \frac{u'(\tilde{c}_t) - \beta R \mathbb{E}_t u'(\tilde{c}_{t+1}) - \mu_t}{\beta R \mathbb{E}_t u'(\tilde{c}_{t+1}) + \mu_t}, \tag{A.27}$$

$$\tau_t^v = \frac{zF_v(1, h_t, v_t) - p_v}{p_v} - \frac{\theta \mu_t}{u'(\tilde{c}_t)}. \tag{A.28}$$

Since (τ_t, τ_t^v) do not appear in the remaining constraints, the simplified Ramsey problem has the same set of controls as the problem in Definition 5 but has an additional constraint (11).

Guess that (11) is slack in the Ramsey problem. If the guess is true, the Ramsey problem is equivalent to the constrained-efficient problem in Definition 5. By Proposition 4, any constrained-efficient plan of Definition 5 is time consistent and entails the UE allocation. By Proposition 1, the UE allocation satisfies (11). Hence, the guess is verified, and the set of Ramsey plans is equivalent to the set of constrained-efficient plans. Consequently, any Ramsey plan entails the UE allocation, and the corresponding Ramsey policy $\{(\tau_t, \tau_t^v)\}_{t=0}^{\infty}$ defined by (A.27) and (A.28) is time consistent.

By Proposition 4, $\{\mu_t\}_{t=0}^{\infty}$ defined by (27) with $\mu_0 = 0$ is part of a constrained-efficient (and thus, Ramsey) plan. By Proposition 1, the UE allocation satisfies (16) and (17). Evaluating (A.27) and (A.28) at the Ramsey plan and imposing (17) in (A.27) and (16) in (A.28), we obtain

$$\tau_t = \frac{-\mu_t}{u'\left(\tilde{c}_t^{\mathrm{UE}}\right) + \mu_t}, \qquad \tau_t^v = -\frac{\theta \mu_t}{u'\left(\tilde{c}_t^{\mathrm{UE}}\right)},$$

which are (36) and (38) in sequential notation.

B Computation

We approximate all functions with linear splines with knots in a grid $\hat{B} = [\underline{b}, \overline{b}] \subset B$ for each $s \in S$. In all cases, $\overline{b} = 0.15$. When we compute the UE allocation described in Section 5.4, $\hat{B}$ has 1000 logarithmically spaced points with $\underline{b} \approx -11.23$. In this case, the lower bound $\underline{b}$ is approximately 10^{-4} above minus the natural borrowing limit, and the logarithmic spacing helps to account for the greater curvature in the policy functions close to the natural borrowing limit. For all other equilibria, $\hat{B}$ has 60 linearly spaced points with $\underline{b} = -0.3$, which is the same grid as in Bianchi and Mendoza (2018). Let $\hat{X} = \hat{B} \times S$.

B.1 Decentralized competitive equilibrium

The DE of Remark 1 is a set of functions $\{\tilde{c}, h, v, b', q, \mu\}$ that satisfy (9)–(15). It will be useful to note that if $\mu_x = 0$, (11) and (12), given the functional forms (39), imply

$$v_x = v_x^{\mathrm{UE}} \equiv \left[\left(\frac{\alpha_h}{\chi} \right)^{\alpha_h} \left(\frac{\alpha_v}{p_v} \right)^{1-\alpha_h+\omega} z^{1+\omega} \right]^{\frac{1}{(1+\omega)(1-\alpha_v)-\alpha_h}}, \tag{B.1}$$

where v_x^{UE} is the UE level of inputs. Moreover, (11) and (39) imply

$$F_v(1, h_x, v_x) \propto v_x^{-\frac{(1+\omega)(1-\alpha_v)-\alpha_h}{1-\alpha_h+\omega}},$$

which is strictly decreasing in v_x since $(1+\omega)(1-\alpha_v) - \alpha_h \geq 1 - \alpha_v - \alpha_h \geq \alpha_k > 0$ and $1-\alpha_h+\omega \geq 1-\alpha_h \geq \alpha_k+\alpha_v > 0$. Therefore, (12) implies that $\mu_x = 0$ if and only if $v_x = v_x^{\mathrm{UE}}$ and $\mu_x > 0$ if and only if $v_x < v_x^{\mathrm{UE}}$, and thus the complementary slackness conditions (10) and (15) are equivalent to

$$0 = \left(v_x^{\mathrm{UE}} - v_x\right) \left(\kappa q_x + \frac{b'_x}{R} - \theta p_v v_x \right), \qquad v_x \leq v_x^{\mathrm{UE}}, \qquad -\frac{b'_x}{R} + \theta p_v v_x \leq \kappa q_x. \tag{B.2}$$

It follows that the equilibrium system (9), (11)–(14), and (B.2) simplifies to a bivariate system (13) and (14) in $\{b', q\}$, with (B.2), (11), (9), and (12) determining the remaining functions v, h, $\tilde{c}$, and μ, respectively, conditional on $\{b', q\}$ as follows:

$$v_x = \min\left\{v_x^{\mathrm{UE}}, \frac{1}{\theta p_v}\left(\kappa q_x + \frac{b'_x}{R}\right)\right\}, \tag{B.3}$$

$$h_x = \left(\frac{\alpha_h}{\chi} z v_x^{\alpha_v}\right)^{\frac{1}{1-\alpha_h+\omega}}, \tag{B.4}$$

$$\tilde{c}_x = zF(1, h_x, v_x) - p_v v_x - g(h_x) + b - \frac{b'_x}{R}, \tag{B.5}$$

$$\mu_x = \frac{u'(\tilde{c}_x)}{\theta p_v}\Big(zF_v(1, h_x, v_x) - p_v\Big). \tag{B.6}$$

To compute the fixed point $\{b', q\}$, we use an algorithm that resembles both the time iteration (Coleman, 1990) and fixed-point iteration (Judd, 1998, p. 599). Let $\{b^n, q^n\}$ denote the guess for $\{b', q\}$ at iteration $n \geq 1$ and $\{v^n, h^n, \tilde{c}^n, \mu^n\}$ the corresponding $\{v, h, \tilde{c}, \mu\}$ given by (B.3)–(B.6). Proceed as follows.

1. Choose $\epsilon > 0$, $\rho \in [0, 1)$, and a metric $d : \mathbb{R}^{|\hat{X}|} \times \mathbb{R}^{|\hat{X}|} \to \mathbb{R}_+$ over the vectors of spline values at the knots. Set $n = 1$ and define the splines b^1 and q^1 by

$$b_x^1 = b, \qquad q_x^1 = \frac{\beta}{1-\beta} zF_k\left(1, h_x^{\mathrm{UE}}, v_x^{\mathrm{UE}}\right),$$

   for all $x = (b, (z, R, \kappa)) \in \hat{X}$, where h^{UE} is given by (B.4) with $v = v^{\mathrm{UE}}$ given by (B.1).

2. At iteration $n \geq 1$, do the following.

   (a) For each $x = (b, s) \in \hat{X}$, find $\hat{b}_x$ that solves a nonlinear equation

$$u'(\hat{c}(\hat{b}_x, q_x^n, x)) = \beta R \mathbb{E}_s u'(\tilde{c}^n(\hat{b}_x, s')) + \hat{\mu}(\hat{b}_x, q_x^n, x),$$

   where $\hat{c}(\hat{b}_x, q_x^n, x)$ and $\hat{\mu}(\hat{b}_x, q_x^n, x)$ are given by (B.5) and (B.6), respectively, with $(b'_x, q_x) = (\hat{b}_x, q_x^n)$ in (B.3)–(B.6).

   (b) For each $x = (b, s) \in \hat{X}$, set

$$\hat{q}_x = \frac{\beta}{u'(\hat{c}(\hat{b}_x, q_x^n, x))} \mathbb{E}_s\left[u'(\tilde{c}_{x'}^n)\Big(z'F_k(1, h_{x'}^n, v_{x'}^n) + q_{x'}^n\Big) + \mu_{x'}^n \kappa' q_{x'}^n\right],$$

   with $x' = (\hat{b}_x, (z', R', \kappa'))$.

3. If $\max\{d(b^n, \hat{b}), d(q^n, \hat{q})\} < \epsilon$, set $\{b', q\} = \{b^n, q^n\}$, and stop. Otherwise, set $b^{n+1} = \rho b^n + (1-\rho)\hat{b}$, $q^{n+1} = \rho q^n + (1-\rho)\hat{q}$, $n = n + 1$, and go to step 2.

We choose $\epsilon = 10^{-6}$, $\rho = 0$, and the maximum metric $d = d_\infty$.

B.2 Unconstrained allocation

By Proposition 1, the UE allocation of Definition 2 is a set of functions $\{\tilde{c}, h, v, b'\}$ that satisfy (9), (11), (16), and (17). The optimality conditions (11) and (16) determine v and h, with closed-form solutions given by (B.1) and (B.4), respectively. The system reduces to a functional equation (17) in b', with $\tilde{c}$ given by (B.5) conditional on b'. Let b^n denote the guess for b' at iteration $n \geq 1$ and $\tilde{c}^n$ the corresponding $\tilde{c}$ given by (B.5). Proceed as follows.

1. Choose $\epsilon > 0$, $\rho \in [0, 1)$, and a metric $d : \mathbb{R}^{|\hat{X}|} \times \mathbb{R}^{|\hat{X}|} \to \mathbb{R}_+$ over the vectors of spline values at the knots. Set $n = 1$ and define the spline b^1 by $b^1_x = b$ for all $x = (b, s) \in \hat{X}$.

2. At iteration $n \geq 1$, for each $x = (b, s) \in \hat{X}$, find $\hat{b}_x$ that solves a nonlinear equation

$$u'(\hat{c}(\hat{b}_x, x)) = \beta R \mathbb{E}_s u'(\tilde{c}^n(\hat{b}_x, s')),$$

   where $\hat{c}(\hat{b}_x, x)$ is given by (B.5) with $b'_x = \hat{b}_x$. If there is no solution in $\hat{B}$, set $\hat{b}_x$ to the value in $\hat{B}$ that achieves the smallest residual (either $\underline{b}$ or $\overline{b}$). If $\underline{b}$ is small enough (but greater than minus the natural borrowing limit) and $\overline{b}$ is large enough, after a few iterations, there will be exactly one $x \in \hat{X}$ at which there is no interior solution: specifically, $\hat{b}(\underline{b}, (\underline{z}, \overline{R}, \cdot)) = \underline{b}$. (Note that the UE allocation does not depend on κ.)

3. If $d(b^n, \hat{b}) < \epsilon$, set $b' = b^n$ and stop. Otherwise, set $b^{n+1} = \rho b^n + (1 - \rho)\hat{b}$, $n = n + 1$, and go to step 2.

We choose $\epsilon = 10^{-6}$, $\rho = 0$, and the maximum metric $d = d_\infty$.

B.3 Optimal time-consistent policies

The optimal time-consistent policy that implements the UE allocation, described in Section 5.4, is obtained directly from Proposition 8 using the UE allocation functions. The optimal time-consistent policies analyzed in Sections 5.2 and 5.3 are each part of an MPE that, given $\underline{\tau} \leq 0$ and $\overline{\tau} \geq 0$, entails the best-response problem

$$V(b, s) = \max_{\hat{c}, \hat{h}, \hat{v}, \hat{b}, \hat{q}, \hat{\mu}, \hat{\tau}} \left[u(\hat{c}) + \beta \mathbb{E}_s V(\hat{b}, s') \right]$$

subject to

$$
\begin{aligned}
\hat{c} + \frac{\hat{b}}{R} &\leq zF(1, \hat{h}, \hat{v}) - p_v\hat{v} - g(\hat{h}) + b, \\
-\frac{\hat{b}}{R} + \theta p_v \hat{v} &\leq \kappa\hat{q}, \\
g'(\hat{h}) &= zF_h(1, \hat{h}, \hat{v}), \\
\left(1 + \theta\frac{\hat{\mu}}{u'(\hat{c})}\right) p_v &= zF_v(1, \hat{h}, \hat{v}), \\
\hat{q}u'(\hat{c}) &= \beta\mathbb{E}_s\left[u'(\tilde{c}_{x'})\Big(z'F_k(1, h_{x'}, v_{x'}) + q_{x'}\Big) + \mu_{x'}\kappa' q_{x'}\right], \\
0 &= \hat{\mu}\left(\kappa\hat{q} + \frac{\hat{b}}{R} - \theta p_v\hat{v}\right), \qquad \hat{\mu} \geq 0, \\
u'(\hat{c}) &= (1 + \hat{\tau})\Big(\beta R\mathbb{E}_s u'(\tilde{c}_{x'}) + \hat{\mu}\Big), \\
\hat{\tau} &\in [\underline{\tau}, \overline{\tau}],
\end{aligned}
$$

with $x' = (\hat{b}, (z', R', \kappa'))$, for all $x = (b, s) = (b, (z, R, \kappa)) \in X$. An MPE is a set of functions $\{\tilde{c}, h, v, b', q, \mu, \tau, V\}$ that satisfy $(\tilde{c}_x, h_x, v_x, b'_x, q_x, \mu_x, \tau_x) = (\hat{c}_x, \hat{h}_x, \hat{v}_x, \hat{b}_x, \hat{q}_x, \hat{\mu}_x, \hat{\tau}_x)$ and $V(b, s) = u(\tilde{c}_x) + \beta\mathbb{E}_s V(b'_x, s')$ for all $x = (b, s) \in X$.

Computing an MPE simplifies to finding the functions $\{b', q, V\}$, with $\{v, h, \tilde{c}, \mu, \tau\}$ given by (B.3)–(B.6) and

$$
\tau_x = \frac{u'(\tilde{c}_x)}{\beta R\mathbb{E}_s u'(\tilde{c}_{x'}) + \mu_x} - 1, \tag{B.7}
$$

respectively, given $\{b', q\}$. Let $\{b^n, q^n, V^n\}$ denote the guess for $\{b', q, V\}$ at iteration $n \geq 1$ and $\{v^n, h^n, \tilde{c}^n, \mu^n\}$ the corresponding $\{v, h, \tilde{c}, \mu\}$ given by (B.3)–(B.6). Proceed as follows.

1. Choose $\epsilon, \epsilon_V > 0$, $\rho, \rho_V \in [0, 1)$, and a metric $d : \mathbb{R}^{|\hat{X}|} \times \mathbb{R}^{|\hat{X}|} \to \mathbb{R}_+$ over the vectors of normalized spline values at the knots, with the corresponding $\bar{d}$ defined by $\bar{d}(x, y) = d(x./(1 + |x|), y./(1 + |x|))$, where ./ denotes the element-wise division. Set $n = 1$ and the splines $\{b^1, q^1, V^1\}$ to those in the closest computed equilibrium, e.g., $\{b^1, q^1, V^1\} = \{b^{\text{DE}}, q^{\text{DE}}, V^{\text{DE}}\}$ if $\{\underline{\tau}, \overline{\tau}\} = \{-\infty, \infty\}$.

2. At iteration $n \geq 1$, for each $x = (b, s) \in \hat{X}$, solve

$$
\hat{V}(b, s) = \max_{\hat{b}_x, \hat{q}_x}\Big[u(\hat{c}(\hat{b}_x, \hat{q}_x, x)) + \beta\mathbb{E}_s V^n(\hat{b}_x, s')\Big] \tag{B.8}
$$

subject to

$$\hat{q}_x u'(\hat{c}(\hat{b}_x, \hat{q}_x, x)) = \beta \mathbb{E}_s \left[u'(\tilde{c}^n_{x'}) \Big(z' F_k(1, h^n_{x'}, v^n_{x'}) + q^n_{x'} \Big) + \mu^n_{x'} \kappa' q^n_{x'} \right], \tag{B.9}$$

$$\hat{\tau}(\hat{b}_x, \hat{q}_x, x, n) \in [\underline{\tau}, \overline{\tau}], \tag{B.10}$$

with $x' = (\hat{b}_x, (z', R', \kappa'))$, where $\hat{c}(\hat{b}_x, \hat{q}_x, x)$ and $\hat{\mu}(\hat{b}_x, \hat{q}_x, x)$ are given by (B.5) and (B.6), respectively, with $(b'_x, q_x) = (\hat{b}_x, \hat{q}_x)$ in (B.3)–(B.6), and

$$\hat{\tau}(\hat{b}_x, \hat{q}_x, x, n) = \frac{u'(\hat{c}(\hat{b}_x, \hat{q}_x, x))}{\beta R \mathbb{E}_s u'(\tilde{c}^n_{x'}) + \hat{\mu}(\hat{b}_x, \hat{q}_x, x)} - 1,$$

based on (B.7). This bivariate constrained maximization problem can have a nonconvex feasible set and multiple local maxima. (See also Appendix B.4.) To find the *global* maximum, we transform the original problem into a set of bound-constrained univariate maximization problems in $\hat{b}_x$, with $\hat{q}_x$ determined as the *largest* root of (B.9) given $\hat{b}_x$. (See Figure B.6 for an example of multiple roots.) Note that, for a fixed $\hat{b}_x$, and thus $\mathbb{E}_s V^n(\hat{b}_x, s')$, the maximum in (B.8) is achieved at the greatest value of current net consumption $\hat{c}(\hat{b}_x, \hat{q}_x, x)$, and (B.3)–(B.5) imply that net consumption is increasing in the asset price. The maximization involves the following steps.

(a) Through a numerical investigation of the nonlinear equation (B.9), find $\hat{B}^n_x \subset \hat{B}$ such that (B.9) has a root in $\hat{q}_x$ given $\hat{b}_x \in \hat{B}^n_x$. The set $\hat{B}^n_x$ can be disconnected (see Figure B.5), so $\hat{B}^n_x = \bigcup_{i=1}^m \hat{B}^n_{x,i}$ for some $m \geq 1$.

(b) For each $i \in \{1, 2, \ldots, m\}$ and $\hat{B}^n_{x,i} \subset \hat{B}^n_x$, solve the maximization problem in (B.8) in terms of $\hat{b}_x$, with $\hat{q}_x$ determined as the largest root of (B.9) given $\hat{b}_x$. Select the global maximum point $\hat{b}_x$, with the corresponding $\hat{q}_x$ and $\hat{V}(b, s)$.

(c) Check that (B.10) holds at $(\hat{b}_x, \hat{q}_x)$. If not, through a numerical investigation of (B.10), refine $\hat{B}^n_x$ such that it is consistent with (B.10), which may entail shrinking/removing/dividing some subsets of $\hat{B}^n_x$, and then go back to (b).

3. If $\max\{\bar{d}(b^n, \hat{b}), \bar{d}(q^n, \hat{q})\} < \epsilon$ and $\bar{d}(V^n, \hat{V}) < \epsilon_V$, set $\{b', q, V\} = \{b^n, q^n, V^n\}$, and stop. Otherwise, set $b^{n+1} = \rho b^n + (1-\rho)\hat{b}$, $q^{n+1} = \rho q^n + (1-\rho)\hat{q}$, $V^{n+1} = \rho_V V^n + (1-\rho_V)\hat{V}$, $n = n+1$, and go to step 2.

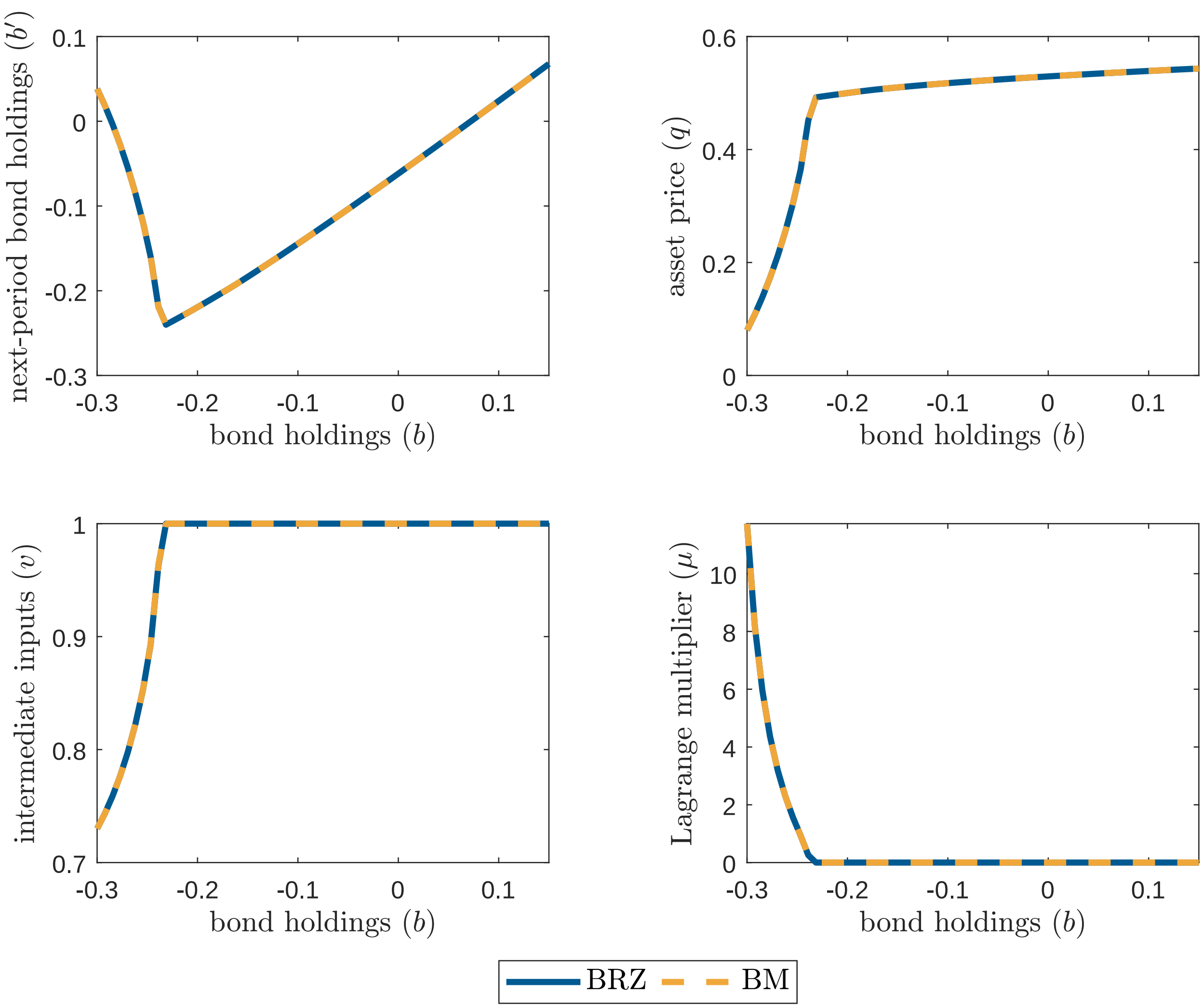
We choose $\epsilon = 10^{-6}$, $\epsilon_V = 10^{-8}$, $\rho = 0.7$, $\rho_V = 0$, and the maximum metric $d = d_\infty$.

B.4 Replicating Bianchi and Mendoza (2018)

The DE allocation we computed is very similar to that in Bianchi and Mendoza (2018) (henceforth "BM"), as illustrated in Figures B.1 and B.2. However, there are significant differences between the SP allocations. Specifically, we find mostly underborrowing rather than overborrowing when the collateral constraint is binding (top-left panel in Figure B.3), and overborrowing is quantitatively smaller when the collateral constraint is slack (top-left panel in Figure B.4), consistent with the fact that the ex post component of the optimal time-consistent policy is critical in this economy. As a result, we obtain twice as large welfare gains from the SP allocation (0.61% of permanent consumption, Table 2) compared to BM (0.30%, as reported in Tables 2 and 3 and can be verified in the replication package).

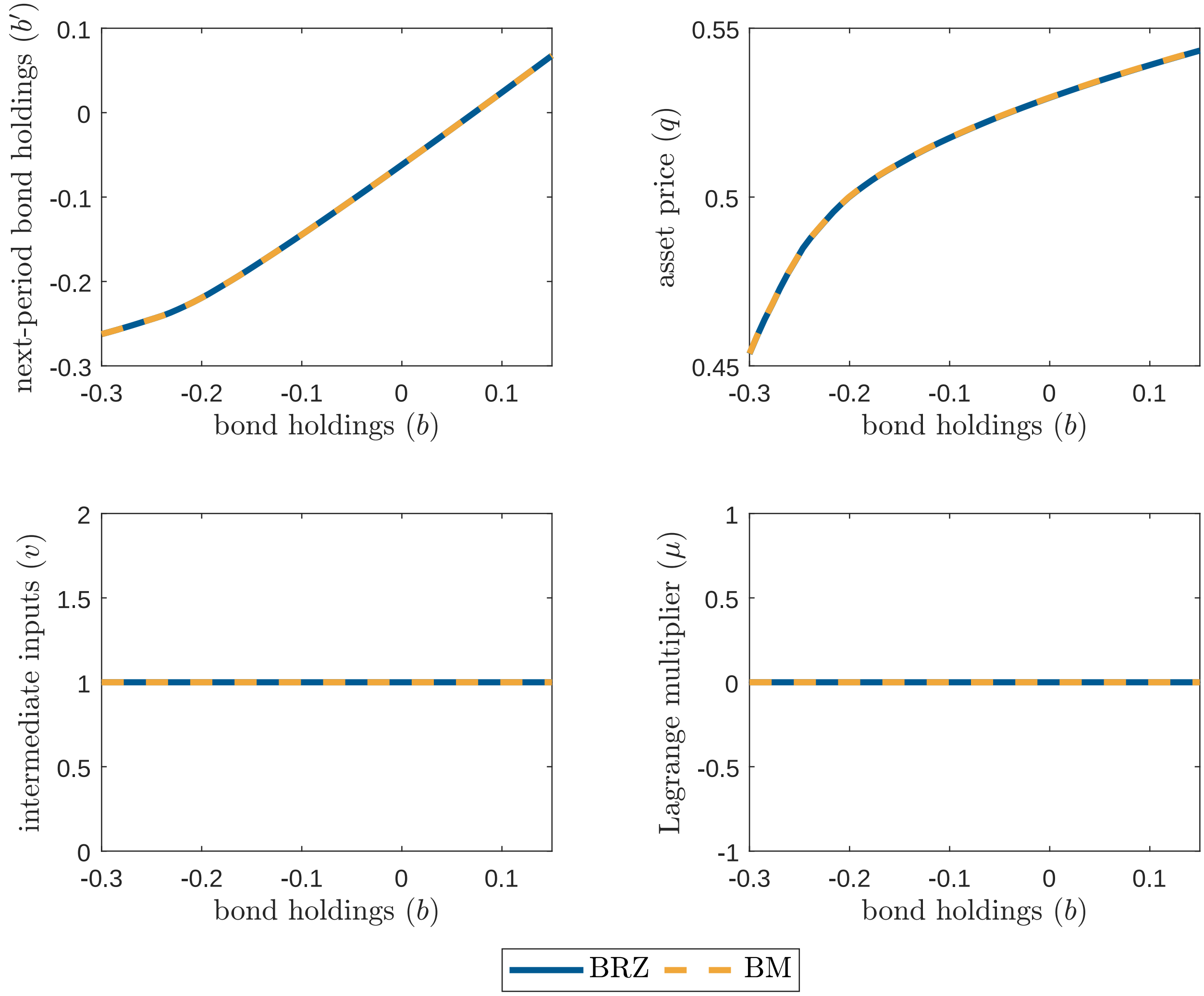
In the correspondence with the authors, we confirmed that these numerical differences can arise because the BM `Fortran` code does not fully account for the nonconvexities of the planner's feasible set. (See also Appendix B.3.) First, as illustrated in Figure B.5, the current planner's objective function in (B.8) can have multiple local maxima, and the `Fortran` routine `mnbrak`, used by BM to bracket the maximum, can bracket a local but not *global* optimum. This can happen when the planner's feasible set is disconnected. In such cases, the global optimum can have significantly more borrowing than a local optimum. Second, as illustrated in Figure B.6, for a given level of next-period bond holdings, if the collateral constraint is binding, there can be multiple pairs of intermediate inputs and the asset price that satisfy (B.3) and (B.9), and the `Fortran` routine `zbrac`, used by BM to bracket the root, can fail to bracket the *welfare-maximizing* (largest) root.

Figure B.1: DE policy functions in the bad state



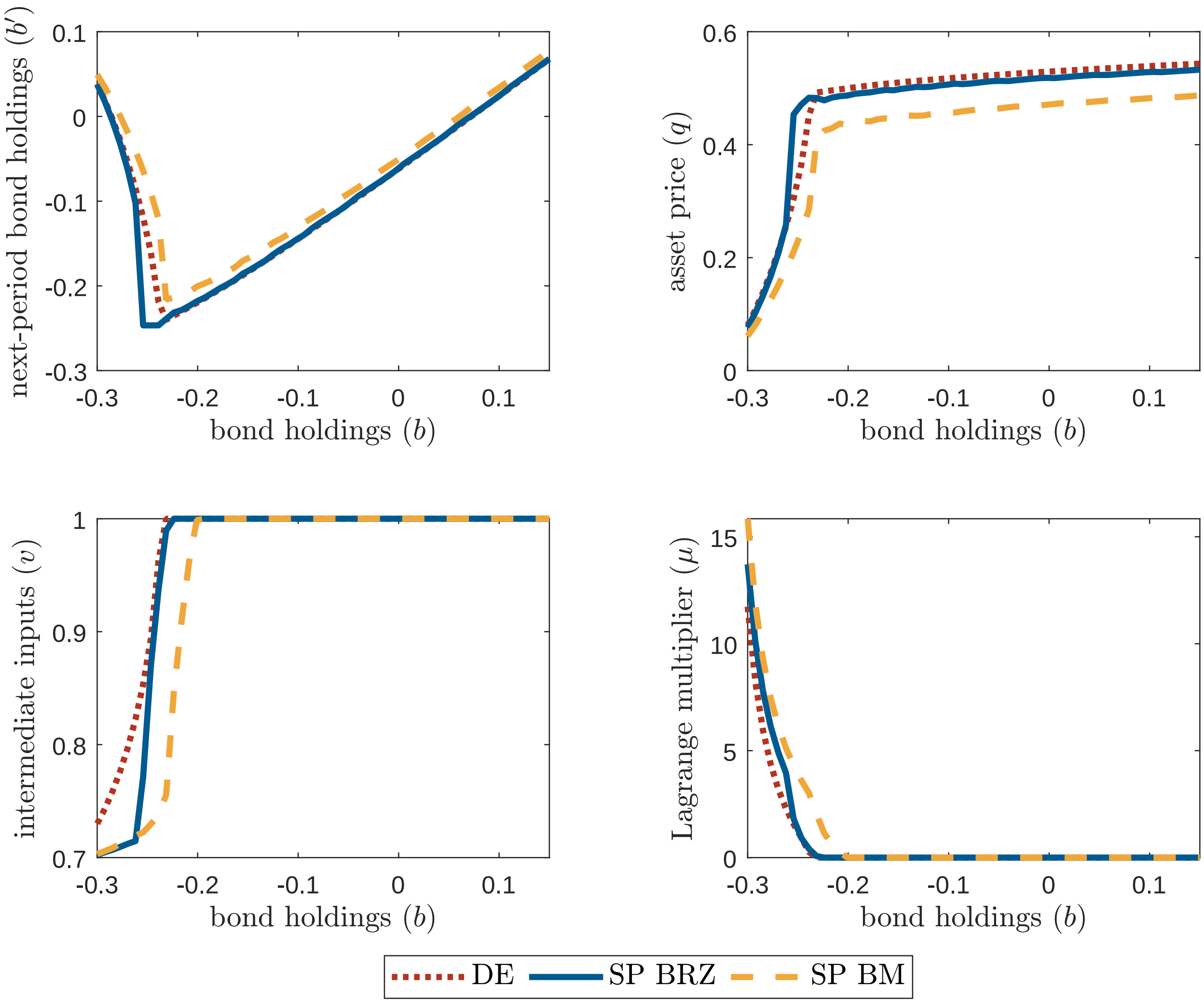
Notes: "BRZ" is our computation, "BM" is the BM replication package, "bad state" = (average z, high R, low κ), as in Fig. 2 and Fig. 3 in BM.

Figure B.2: DE policy functions in the good state



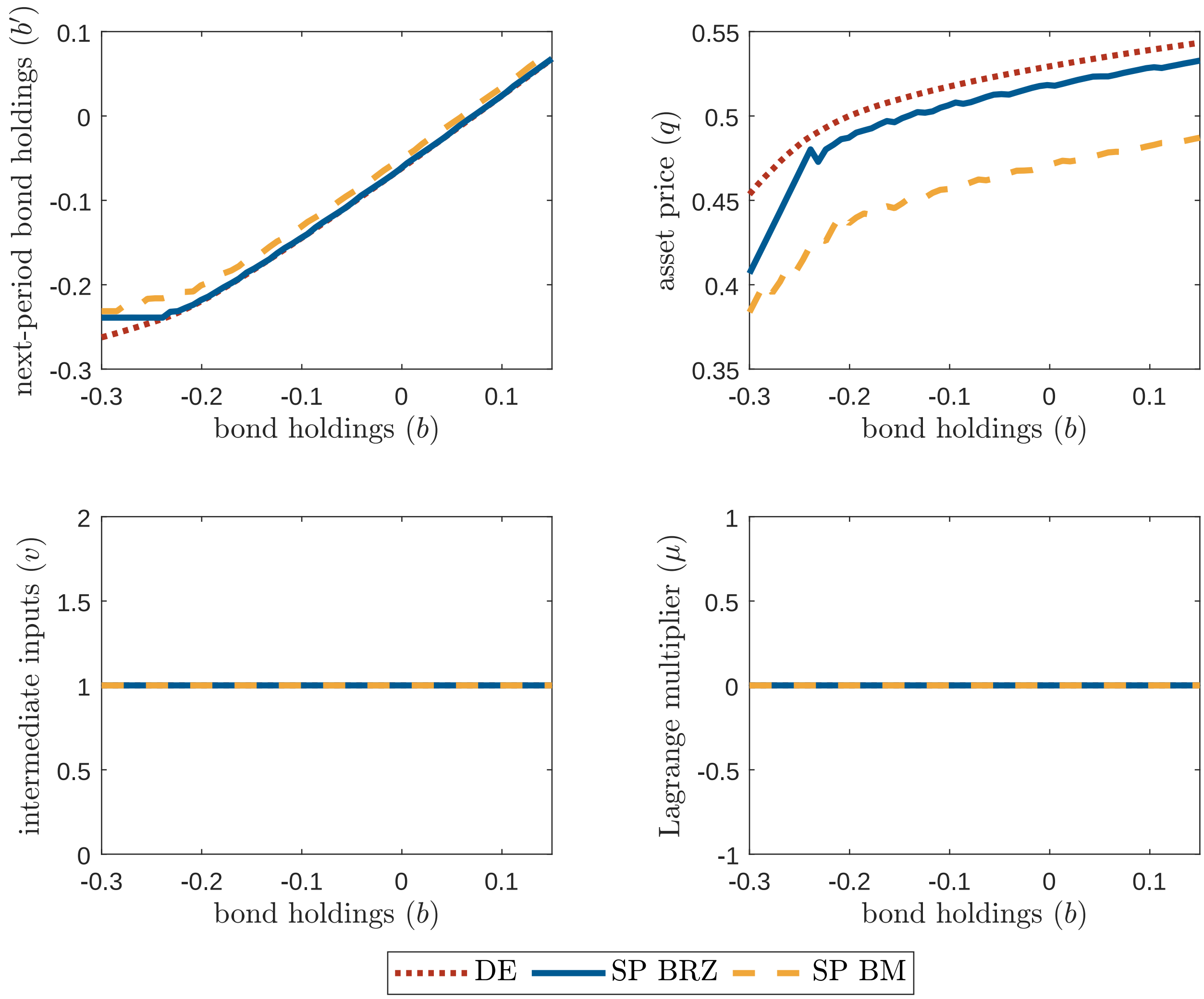
Notes: "BRZ" is our computation, "BM" is the BM replication package, "good state" = (average z, high R, high κ), as in Fig. 5A in BM.

Figure B.3: DE and SP policy functions in the bad state



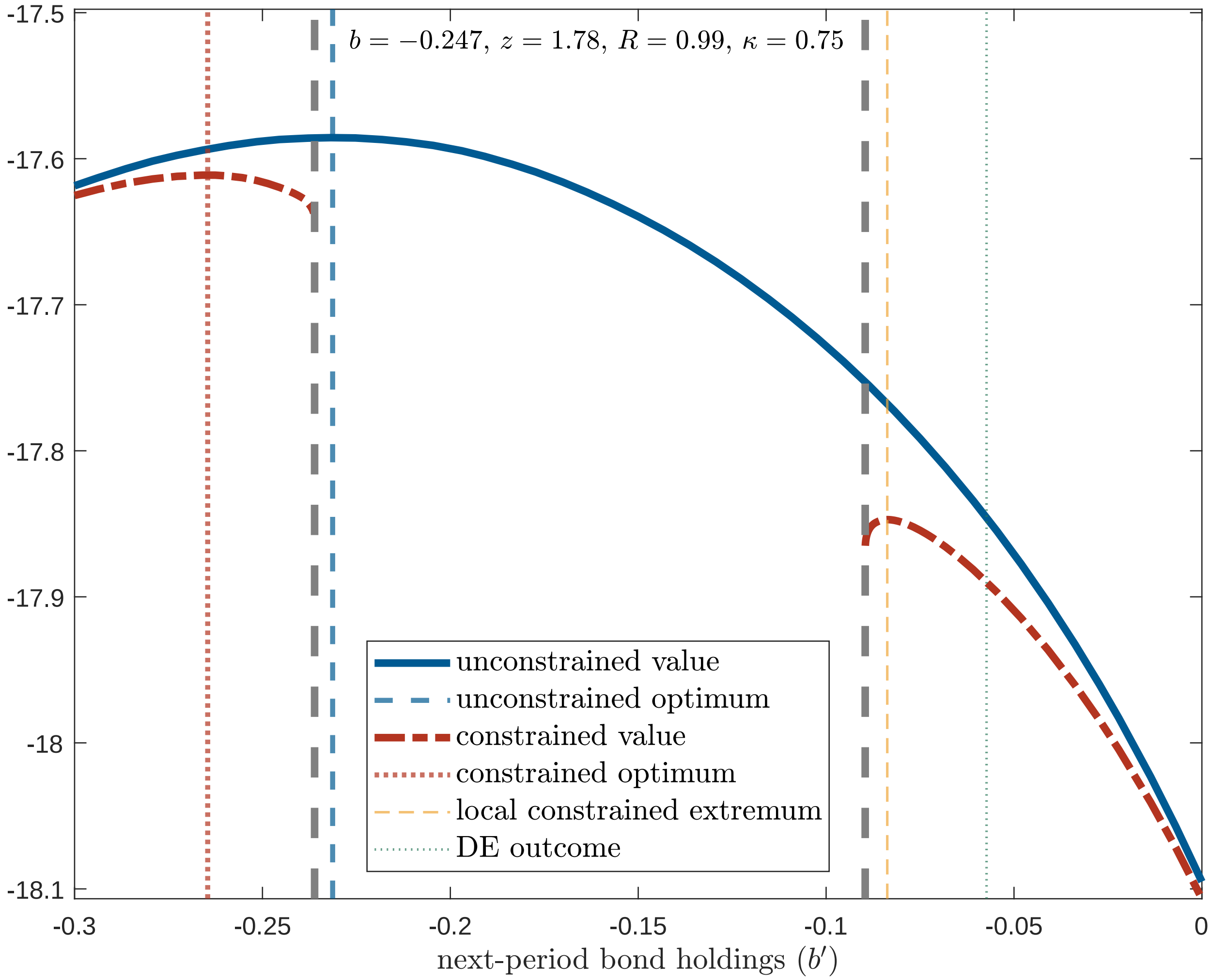
Notes: "DE" and "SP BRZ" is our computation, "SP BM" is the BM replication package, "bad state" = (average z, high R, low κ), as in Fig. 2 and Fig. 3 in BM.

Figure B.4: DE and SP policy functions in the good state



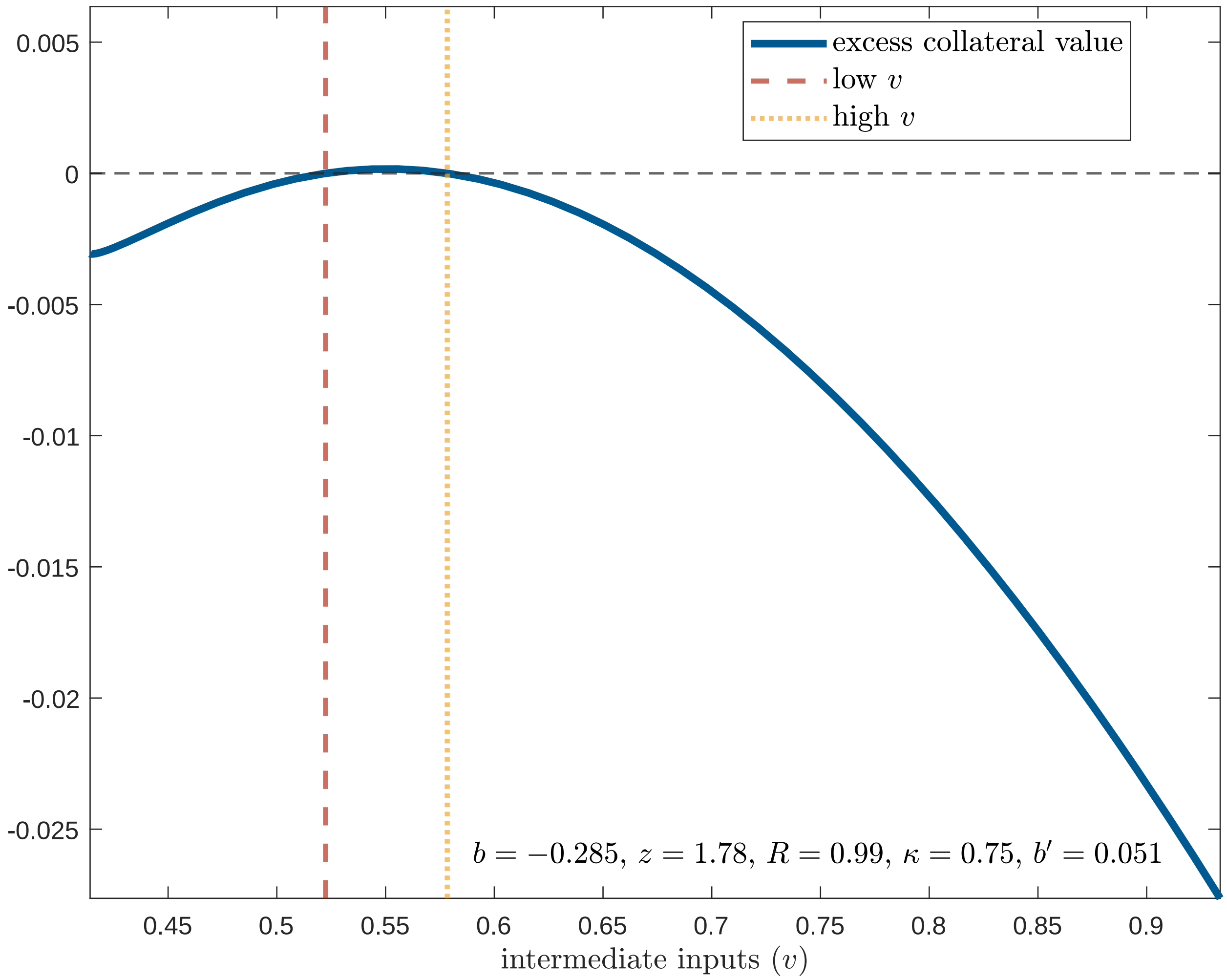
Notes: "DE" and "SP BRZ" is our computation, "SP BM" is the BM replication package, "good state" = (average z, high R, high κ), as in Fig. 5A in BM.

Figure B.5: SP best response to DE with nonconvex feasible set and multiple local extrema



Notes: The "unconstrained" and "constrained" values correspond to the planner's objective function in (B.8) considered as a function of next-period bond holdings $\hat{b}_x$. (See Appendix B.3.) These values are conditional on the state $x = (b, (z, R, \kappa)) = (-0.247, (1.78, 0.99, 0.75))$ and next-period policy functions being the DE policy functions. The "unconstrained value" ignores the collateral constraint by setting $\hat{v}_x = v_x^{\mathrm{UE}}$ in (B.3): in this case, any $\hat{b}_x \in [-0.3, 0]$, including the "unconstrained optimum," violates the collateral constraint. The feasible set is nonconvex (disconnected): if $\hat{b}_x$ is between the two dashed gray vertical lines, (B.9) does not have a solution in $\hat{q}_x$ given $\hat{b}_x$. The "constrained value" accounts for the binding collateral constraint. If one were to use the bracketing routine `mnbrak`, starting from the neighborhood of the "DE outcome," as in the BM code, it would bracket the suboptimal "local constrained extremum" instead of the "constrained optimum."

Figure B.6: SP problem with multiple constrained intermediate inputs



Notes: The "excess collateral value" $\kappa \hat{q}_x + \hat{b}_x/R - \theta p_v \hat{v}_x$ is the binding collateral constraint residual as a function of inputs $\hat{v}_x$, conditional on the state $x = (b, (z, R, \kappa)) = (-0.285, (1.78, 0.99, 0.75))$, next-period bond holdings $\hat{b}_x = 0.051$, next-period policy functions being the DE policy functions, and the asset price $\hat{q}_x$ given by (B.9). If one were to use the bracketing routine `zbrac`, starting from the neighborhood of the UE level of inputs $v_x^{\mathrm{UE}} \approx 0.93$, as in the BM code, it would fail to bracket the roots. The largest root ("high v") is welfare-maximizing. Note that here, to achieve a greater similarity with the BM code, we are solving for inputs (instead of the asset price as in Appendix B.3) conditional on next-period bond holdings. Given $(\hat{b}_x, v_x)$, (B.4) and (B.5) give $\hat{c}_x$, and (B.9) gives $\hat{q}_x$ in closed form, since the right-hand side depends only on $\hat{b}_x$. These two approaches are equivalent: conditional on next-period bond holdings, (B.3) implies a one-to-one correspondence between inputs and asset prices when the collateral constraint is binding. When the collateral constraint is slack, inputs are at the UE level, consumption does not depend on the asset price, and the latter is pinned down uniquely by (B.9).